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DESIGN AND ANALYSIS OF FIXED TIME RELIABILITY DEMONSTRATION TESTS

J. Adelsberg

Naval Air Development Center
Warminster, Pennsylvania

1 November 1975

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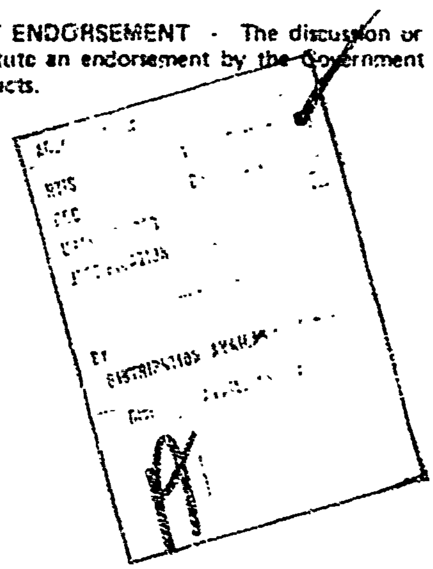

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The principal reliability demonstration test methods are identified and their characteristics reviewed. The use of a graphic procedure for the design and analysis of fixed time tests is discussed and examples illustrating application to the design of Classical, Bayes and Hybrid plans are given. The graphic procedure is also applied to evaluate the effect on the Bayes posterior risks as a function of different prior distributional characteristics and to make comparisons between the different test methods. A technique for

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constructing the operating characteristic (O.C.) curve for any fixed time test is presented and the O.C. data for various test plans are tabulated.

SUMMARY

INTRODUCTION

Application of the classical demonstration method tends to lead to long tests when the mean time between failures (MTBF) of the device to be tested is large and few samples are available. The Bayes technique while often successful in shortening test time requirements, introduces other difficulties. The most significant is that test properties generally are quite sensitive to the characteristics of the prior distribution. When the prior assignment is representative of the MTBF characteristic of the device to be tested there is considerable gain, in terms of test costs, resulting from the use of the Bayes approach. However, in most instances the needed data and data accuracy is lacking, leading to uncertain results.

Attempts to mitigate this problem have led to the development of yet another technique, denoted as the Hybrid Method. Here prior information as well as a classical criterion is utilized in test design, usually in the form of a Bayes producer's risk and a classical consumer's risk. This combination results in a number of attractive test features. An important one, especially from the users point of view, is that the consumer protection is independent of the prior distribution and remains within a specified level. The producer also realizes some benefit in that he is able to incorporate pertinent information, acquired during development, into the test plan design and thus affect test properties such as producer's risk and test time. Another attribute that is of mutual benefit is that a Hybrid test can generally be performed in a shorter time period than a comparable classical plan. The exact savings in test time depend on the characteristics of the prior distribution and the specific test parameter values selected, but can be readily determined with the aid of the charts provided in this report.

A primary aim of this study was to investigate some consequences of the Bayes method, especially those aspects of the procedure that impact on the user, i.e., the Government. Concern had been voiced about the consumer protection provided by these plans and there was apprehension that this approach seemingly led to the acceptance of bad material, much in excess of allowable errors. Recent applications of Bayes tests in which test time was extremely short or completely eliminated tended to reinforce this feeling. The crux of the problem was the lack of a quantitative technique to assess the impact of the prior information on consumer related test properties. Thus, an important element of this study concerned the development of a method capable of evaluating these dependencies for different choices of prior information.

A further objective was to develop procedures to enable the design and analysis of fixed time test plans, for all currently used methods, to be accomplished in an efficient manner. This objective was set forth

because of the practical difficulties that presently exist in the implementation of a test requirement, especially when attempting to use the newer techniques. The pertinent literature often lacks the needed depth or detail to permit direct application to test design. The intention, therefore, was to provide a compendium of user oriented material suitable for this purpose.

In spite of these seemingly diverse objectives, solutions were derived using a common approach. The key element in this approach is a graphical procedure that proved to be as versatile when applied for purposes of test plan design as it was in the analysis of these plans. For example, with the aid of these graphs, it became a relatively simple matter to compute the change in Bayes consumer's and producer's risk resulting from adjustments in the prior parameters while keeping other variables fixed. The graphical procedure also performed well when utilized as a design tool. It provided a simple, accurate, yet flexible technique in the construction of any fixed time test plan, regardless of method.

Finally, the availability of alternate test methods, while introducing additional design flexibility, may actually complicate the task of test plan selection unless some logical and systematic scheme is employed for this function. To help guide this effort, the report provides a set of criteria and explains how they can be applied to a derive a preferred test plan.

CONCLUSIONS

a. A Bayes test is a relatively poor method for detecting inadequate devices especially when the prior is optimistic.

b. Use of an optimistic prior in a Bayes test will generally result in a short test, but it will also give rise to a dramatic reduction in user protection (i.e., the protection against acceptance of inadequate devices.)

c. Since the formal test period in a Bayes plan with an optimistic prior is invariably much shorter than the prior MTBF estimate, it represents a negligible factor in the test decision and its use should therefore be discontinued (i.e., the major benefit of a test of this type is of a psychological rather than a statistical nature).

d. The suggested use of the Probability of Acceptance, $P(A_{cc})$ as the producer's criterion in a Bayes test in lieu of the Bayes posterior producer's risk does have merit in that it represents a more meaningful criterion for a contractor. However, construction of Bayes tests using this criterion tends to lead to very short tests with attendant loss of consumer protection and thus should be avoided. $P(A_{cc})$ can, however, be constructively used in another manner; it can be employed as a technique for predicting the reliability status of a product in the various stages of its development and thus help to flag problems that may require corrective action.

e. More wide spread application of the Bayes approach depends mainly on the development of more realistic priors. One way this can be achieved is through implementation of a reliability growth estimation procedure. The test data generated during systems development and utilized, in part, to satisfy the requirements of tracking reliability growth as the design progresses, could also serve as the basis for constructing more representative priors. (Instituting requirements for reliability growth modeling would not only achieve much better control of the reliability of an evolving system, but would also provide a representative data base for Bayes type demonstration tests.)

f. At present, the Hybrid test method represents the most cost effective solution to a broad spectrum of fixed time demonstration test problems and should therefore be specified and applied more frequently.

RECOMMENDATIONS

a. The portion of MIL STD 781-B dealing with fixed time tests should be updated to incorporate use of the graphical design and analysis procedure for the construction of classical, Bayes and Hybrid demonstration test plans.

b. Prior to the selection and specification of a test plan, the operating characteristics (O.C.) data of the proposed plan should be generated and examined to ensure that the test plan's performance is consistent with desired test objectives (the technique described in the report may be used to obtain the O.C. data).

c. A program aimed at familiarizing user groups with the characteristics and application of the Hybrid test method should be instituted.

d. The following additional work effort, limited to tasks which are intended to broaden the range of application of already developed techniques, is recommended.

i) Extend the Bayes approach to accommodate applications where reliability instead of MTBF is the appropriate success criterion. Generally this applies to systems that operate over relatively short time periods such as sonobuoys or missiles. This effort would produce results similar to what is available for the MTBF case, but start with a more suitable prior distribution.

ii) Evaluate the sensitivity of the Bayes posterior risks as a function of the parameter values of the prior distribution. Results of this study will provide guidelines on how coverage of a range of parameter values can be achieved with a suitably small number of graphs.

iii) Develop a technique for constructing the Operating Characteristics (O.C.) curve for Bayes sequential tests. A procedure for determining the

O.C. curve for fixed time tests is given in this report. While accomplishment of a comparable capability for sequential tests is more complicated, its availability will permit the performance of a given sequential test to be examined and thus lead to improved test design.

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PREFACE

Reliability demonstration testing is generally regarded as a significant element of a reliability program since it: (1) identifies and helps the design team to focus on a reliability requirement throughout systems development and (2), represents the principal method for verifying whether a given requirement has been achieved in the design. Its positive role notwithstanding, the procedure is often not utilized in systems procurement mainly because of schedule demands or cost considerations. Part of the problem arises from the fact that the principal demonstration method in use today, generally referred to as the "Classical" method, tends to require long tests, especially in cases where the inherent reliability of an item is large.

In contrast with the classical method the Bayes approach makes use of prior information about the possible reliability values of the system to be tested. This information is incorporated into the test design, usually in the form of a density function. The test decision is now based on the prior data as well as the data generated during performance of the test. When the prior distribution is amenable to a relative frequency interpretation, it can be viewed as pseudo test data which, when combined with actual data, is instrumental in effecting the relatively short tests experienced with this method. In most instances, however, development of a prior is based on subjective information which causes great difficulty in assessing the reasonableness and consequences of the various ssumptions made. This is of considerable importance since errors in the estimates of the prior parameters are propagated to the posterior estimates, on which decisions are based.

The Hybrid method combines some of the better features of the other methods. It represents a practical option when the prior assignment is considered to be unacceptable yet use of an efficient test is indicated. Since the consumer protection is stated in terms of the familiar classical consumer's risk, it is independent of the prior information. The prior, however, still plays an important role in that it affects test time and the producer's risk.

The three methods identified represent the range of alternatives currently available in the area of fixed time tests. Given a set of test objectives, selection of suitable test plan requires knowledge of the characteristics of the different methods and a capability to construct and compare competing plans. The principal aim of this report is to convey this information in simple, user oriented terms.

SECTION II

I. SUMMARY OF CHARACTERISTICS OF DIFFERENT TEST METHODS

A. General

While the risks of any Classical fixed time test plan are identical in definition, this is not true for the Bayesian method where, because of lack of agreement about what is meant by consumer's and producer's risk, three different producer's risks and two consumer's risks are utilized at present, (1), (3), (5). Since each producer's risk can be combined with a consumer's risk to form a feasible test plan, it is possible to construct many different Bayes tests. In addition, each Bayes risk can be combined with a Classical risk creating additional test alternatives. The full range of possibilities can perhaps be most easily seen when expressed in the matrix form shown below:

MATRIX OF POSSIBLE TEST METHODS

	1	2	3
1	$\alpha-\beta$	$\alpha-B$	$\alpha-B_1$
2	$A-\beta$	$A-B$	$A-B_1$
3	$A_1-\beta$	A_1-B	A_1-B_1
4	$P(Acc)-\beta$	$P(Acc)-B$	$P(Acc)-B_1$

the symbols denote the following: (A more formal definition of the risk terms is given in the next section.)

α = classical producer's risk

β = classical consumer's risk

A = Bayes posterior producer's risk

B = Bayes posterior consumer risk

A_1 = Average Bayes producer's risk

B_1 = Average Bayes consumer's risk

$P(Acc)$ = Probability of test acceptance (A test criterion used in Bayes tests usually in lieu of the A or A_1 criterion)

$\alpha-\beta$ = designates the Classical consumer's and producer's risk combination

$A-\beta$ = designates the Hybrid criteria utilizing a Classical consumer risk and a Bayesian producer risk

Partitioning of the matrix groups the array in terms of the different test methods. Thus element (1,1) of the matrix represents the Classical (α-β) method. Elements (2,2), (2,3), (3,2), (3,3), (4,2), (4,3) comprise the Bayes method. The other elements contain mixed criteria and are, by definition, Hybrid tests. Not all Hybrid tests are considered to be of equal significance in that those combinations appearing in the first column (i.e., the ones utilizing a classical consumer's risk) are regarded as more practical than those with a classical producer's risk (i.e., elements 1,2 and 1,3 of the matrix). This is due to the assumption that a Hybrid plan is preferred in cases where difficulty is experienced in deriving an acceptable prior. In these situations, it is felt, the users interest would be better served with a test where his risk is expressed in terms of a classical risk. Plans which reverse this condition (i.e., provide a classical producer's risk) were assumed to have lesser practical significance and are not investigated further.

There are a number of assumptions common to all test methods. The most general is that the interarrival times of failure are considered to be independent and identical, exponentially distributed, random variables. The mean of this distribution θ, is the mean life. A direct consequence of this assumption is that the hazard rate is constant, independent of operating time. Another is that the mean time between failures (MTBF) is independent of the number of failures observed and equals the mean life. That is, the mean time between the $(n-1)^{st}$ failure and the n^{th} failure is the same for all n (n = 1, 2, . .N). If the device is repairable then completion of a repair results in an "as good as new" condition. Or, if the device consists of components which are replaced upon failure, the device is considered to be "as good as new" after each replacement. (The term 'device' refers to that particular level of a system's hierarchal structure for which an assumption of a constant hazard rate is considered reasonable. Thus the term 'device' could signify a single component, a series of components or a complex assembly.)

There also exist commonalities in test methodology. For all methods reliability demonstration is a form of hypothesis test whose aim is to distinguish whether $\theta \geq \theta_1$ or $\theta < \theta_1$, where θ_1 is the value of MTBF that is to be demonstrated. All test procedures also call for operating the device for T hours, repairing or replacing it upon failure (in cases where repair is performed average repair time (MTTR) is assumed to be much smaller than the MTBF) and counting the number of failures occurring in time T. If the number of failures is less than r^*, where r^* is the allowable number of failures specified by the test, the decision is made that $\theta \geq \theta_1$ (i.e., θ_1 has been demonstrated). If the number of failures exceeds r^* the decision is that the $\theta < \theta_1$. The test is uniquely determined once T, r^* are chosen. Explicit specification of T, r^* are characteristic of a fixed time test. Tests where T and r^* are not fixed, i.e., sequential tests, are not considered here. In addition to the shared properties, each method also has a number of distinct features such as:

B. Classical Method

The probability of device acceptance depends on the actual value of θ and is denoted by $P(Acc/\theta)$. The curve obtained when plotting this

probability against θ is called the operating characteristic (O.C.) curve. The O.C. curve shows how the chance of acceptance varies with different possible values of θ and is an important quality characteristic of the test. Classical test design generally involves choosing a value of θ, θ_0, and a value α (called producer's risk) and requiring that $P(Acc/\theta_0) \geq 1-\alpha$; and choosing a value of $\theta, \theta_1 < \theta_0$ and a value β (called consumer's risk) and requiring that $P(Acc/\theta_1) \leq \beta$. If a device is accepted it is said that mean life θ_1 has been demonstrated with, at least, confidence $1-\beta$. The use of the parameter θ_0, while not a part of the demonstration requirement, is necessary to control the shape of the O.C. curve for values $\theta \gg \theta_1$, and thus to protect the producer against the use of plans which have a high probability of rejecting devices which more than meet the requirement. If all devices tested are of quality level θ_0, the fraction α would be rejected by the test. Similarly, if all devices have quality level θ_1, the consumer will wind up with 100% marginal devices even though only β percent are accepted. This test method assumes that all devices on test have the same, but unknown, MTBF, and that if future production is accepted on the basis of these tests, they will also have the same MTBF. The Classical method considers MTBF as an unknown parameter and no use is made of prior information about the possible values of MTBF.

C. Bayesian Methods

Here prior information and actual data, each expressed in a specified form, is used in test design. The mixing of the prior and the observed data is accomplished using Bayes theorem and results in a posterior function which reflects the impact of the data on the prior. The mathematics of the mixing process become simplified and interpretation of results facilitated if the prior and the test data are chosen as conjugate functions. This leads to a posterior of the same form as the prior with parameters that are additive constants of the prior parameters. In terms of the demonstration problem of concern here, where MTBF is the figure of merit and failure data is generated in accordance with a Poisson process, the appropriate prior is the inverted gamma density. As a consequence, the posterior is also an inverted gamma density with parameters which are the sum of the prior parameter values and the test data.

While the mechanics of the mixing process involve well defined and noncontroversial operations, they can't be implemented without assignment of a prior distribution. It is this aspect of the Bayesian method that has been the subject of great controversy. Much of it centers on the nature and interpretations of $g(\theta)$, the prior distribution. As pointed out in reference (1), two models generally apply. The first is where θ is assumed to vary from experiment to experiment, according to $g(\theta)$; that is, θ is assumed to be a random variable having a fixed distribution $g(\theta)$ and the values of the MTBF at different times are independent realizations of this random variable, which is not directly observable. In this model, the posterior distribution pertaining to a particular experiment cannot be used as the prior for the next experiment simply because this posterior is not the one generating the next value of θ. As a consequence, no information in the form of a posterior distribution can be carried over from test to test. The use of the

posterior distribution from one test as the prior for the next builds in the assumption that the MTBF at different times is identical and that the additional data pertains to the same MTBF. This, of course, is a different model; one which treats MTBF as a fixed but unknown constant and its distribution $g(\theta)$ represents the degree of belief associated with the possible values of θ. Here MTBF can be considered as a random variable only once and thereafter remains fixed. This means that additional information is gathered about the unknown parameter value as more data becomes available, as happens in sampling distributions. Both models are identical in that not only the form of the prior distribution but the specific distribution is assumed to be known initially.

These assumptions do not pertain to the Empirical Bayes method. In this approach, the data accumulated in performing repetitive tests is used to estimate the prior distribution, which is amenable to a relative frequency interpretation. Investigation of the asymptotic behavior of the Empirical Bayes procedure in reference (2) indicates that after observance of a considerable number of repetitions the Bayes risk for the next test is almost the same as if the prior were known. This is a very desirable property in that nothing need be assumed about the prior, but unfortunately requires a long sequence of tests, of a repetitive nature, before this property is realized. The latter constraint makes it difficult to apply this method to many reliability demonstration problems and it is therefore not discussed further.

As noted previously, several different sets of risk criteria are utilized in the Bayes approach. The three commonly used sets which are also applied in references (3), (4), (5), are:

1) $P(\theta \geq \theta_0/\text{Reject}) = A$ and $P(\theta \leq \theta_1/\text{Accept}) = B$ where A, B denote the posterior producer's, consumer's risk, respectively.

2) $P(\text{Reject}/\theta \geq \theta_0) = A_1$ and $P(\text{Accept}/\theta \leq \theta_1) = B_1$ where A_1, B_1 denote the average Bayes producer's, consumer's risk, respectively.

3) $P(\text{Acceptance}) = A_2$ and B or B_1 where A_2 is the a priori probability of acceptance before the test is conducted. It is used here as a producer's criteria in lieu of A or A_1.

The third combination uses overall acceptance rate for the producer's risk. It has been suggested in references (3), (6), that this quantity has more significance to a producer than either A or A_1 and therefore represents a more pertinent criterion than either of the other expressions. Assuming that rejected devices cost a producer money because of the need to scrap or overhaul them, it does appear doubtful that he would derive comfort from the fact that only, say, 10% of the rejected units have a $\theta \geq \theta_0$ when his overall rejection rate is high. This does suggest that he may prefer to limit the overall rejection rate or to control the probability of having good units rejected. To accomodate these goals a number of the test plans utilizing the A_2-B and the A_1- B_1 criteria sets have been constructed and are examined in the report.

Irrespective of the particular set of risks selected, a Bayes test cannot be formulated without use of a prior distribution. Once the prior has been chosen the test can be designed to limit the risks to remain within selected values under the assumption that the prior distribution is an accurate representation. This point should be kept in mind when use of a Bayes procedure is contemplated. In addition, two other potential difficulties are:

1) If the assumed prior is grossly in error, especially if it is overly optimistic about the capabilities of the device - this procedure can be very poor in detecting inadequate devices. (This is bad for the consumer.)

2) It may be extremely difficult to construct a prior which accurately reflects the expected MTBF capabilities of a device using information gathered during its design and development, or even from past data deemed "suitable" for this purpose.

D. Hybrid Test Method

As pointed out, a serious problem mitigating the application of Bayes tests is the uncertainty associated with the prior distribution. Often a contractor has a prior distribution which puts a great deal of weight on values of $\theta >> \theta_1$ and the user is unwilling to accept this prior. If he does use this prior for the purpose of devising a test he finds that he has uncomfortably large probabilities of accepting devices with $\theta < \theta_1$ and does not share the producer's optimism regarding the small chance that such θ's will be encountered. He recognizes that without having a prior distribution he can believe in, he cannot hope to achieve the aim of risk B, the Bayes posterior consumers risk, regardless of its potential attractiveness. He also realizes that using the B_1 criterion, even with reasonably small $P(\text{Accept}/\theta < \theta_1)$, he can have many poor devices on his hands if all devices tested are, in fact, of poor quality. He therefore prefers to be protected in the classical sense, in terms that he is more familiar with. To accomodate this point of view, a number of Hybrid plans, combining a classical consumer and Bayes producer risk, are developed and discussed in this report. As will be seen, these plans have a number of attractive features.

SECTION III

I. GRAPHIC PROCEDURE

A. General

When several test alternatives are possible, identification of a preferred plan is usually achieved by repeated application of a test design procedure. The characteristics of the procedure therefore, determine how efficiently this task can be accomplished.

For the classical test method, the test selection problem is simple since test alternatives are available in tabulated form and appear in publications such as MIL STD 781B. Selection of an appropriate plan merely involves scrutiny of the tables to identify that set of parameters best suited to meet the given requirements. Of course not all feasible combinations of test parameters are tabulated. If a particular combination not listed is of interest the user has a choice of either selecting a set that comes closest to meeting his goal from the available tabulations or he can derive his own plan by solving a set of simultaneous equations appropriate to the Classical test method. The latter scheme requires some additional effort but can be readily accomplished.

For the Bayes method, tabulations are now beginning to appear, reference (6). Since each plan must include specification of parameter values for the prior distribution, in addition to the usual indices, sizeable tabulations result. For the inverted gamma prior distribution commonly encountered parameter values vary over a 5 to 1 range for each of two parameters. This gives rise to a 25 fold increase in the number of tables required compared to the Classical method(assuming integer parameter values). Since manipulation of large amounts of data tends to become quite awkward and time consuming, this approach is not considered to represent an effective method for test plan selection.

The graphical test design technique described in this report was originally developed to facilitate analysis of Bayes test plans. Its availability eliminates the need for extensive tabulations and removes some of the previously mentioned limitations in regard to choice of test parameter values. While applicable to all test methods the Classical method, because it is the simplest and most familiar technique, is used as a vehicle to explain the characteristics of this procedure and to illustrate the manner in which it can utilized.

B. Classical Test Plans

The graph shown in figure 1 is a computer plot of the risk functions for the Classical method. The graph consists of two sets of curves labelled R*A and R*B. The R*A set depicts the relationship between producer's risk and normalized test times, T/θ_0. A separate curve is drawn for each of the 11 values assigned to R*A. R*A denotes the maximum number of failures allowed in a test for an accept decision. Values

of R*A are incremented in unit steps and vary between zero and ten. Similarly, the R*B set describes the functional beh--ior of consumer's risk, β, with norma' ed test time T/θ_o. Again eleven separate curves are shown, each for different R*B value, as indicated in that figure.

All the information needed to formulate, evaluate and compare a variety of different fixed time Classical tests can be abstracted from these curves. Before giving instructions for doing this, it is informative to examine some of the relationships shown in the graph. As can be seen, all consumer's risk curves tend asymtotically towards zero with increased test time. Since consumer's risk, by definition, is the probability of accepting marginal devices, its value is large for small test times because even bad units ($\theta \leq \theta_1$) will often not fail in a relatively short time. As test time increases the test becomes more discriminatory in that it is able to reject more bad units and consequently, the consumer's risk will decrease. For large T/θ_o values the test becomes rather severe in the sense that only a few failures are allowed over a relatively long test period, even for the largest R*B value indicated. This means that many good units ($\theta >> \theta_1$) as well as bad ones, will be rejected by the test.

Using the definition of producer's risk and applying similar reasoning, it is expected that for small test times α will be small since few units, good or bad, will be rejected. Conversely, for larger test times α will be large due to the severity of the test. When T/θ_o is kept constant, α can be seen to vary inversely with R*A. This result can be explained by noting that conditions for acceptance have been relaxed when the value of R*A is increased, while T/θ_o remains fixed. Consequently the proportion of rejected units will be smaller and therefore the subset consisting of the fraction of rejected units that are good will also be smaller. Depicting the pertinent functions in graphical form permits ready observation of their behavior and thus promotes an understanding of important relationships.

As pointed out previously, the graphs of figure 1 can be used as a simple, accurate, yet flexible test design tool. While the application sequence will vary in accordance with the specific requirements of a given problem, the cases generally encountered can be classified into 3 groups. In the following discussion, sample problems of each group are postulated together with a step by step explanation of how the respective solutions may be obtained.

1. Plans having equal risks and a constraint on the common risk value.

Example 1: Develop test plans having equal α and β values. For the purpose of this example limit α, β≤20%.

The values of T/θ_0, and the maximum allowable number of failures, r* needed to completely specify this test can be obtained from the points of intersection of equal valued R*A and R*B curves. Since valid test conditions require a single value for the allowable number of failures,

R*A must equal R*B. The point of intersection of the R*A = R*B = 10 curve gives a value of 12.8% for equal α, β. The corresponding test time can be determined by drawing a vertical line from the point of intersection to the horizontal (x) axis. The value obtained is approximately 7.4 Θ_0 hours. If five failures are allowed, test time is decreased to approximately 4.0 Θ_0 hours but the risks increase to 20.4%. The latter plan represents the lower limit for allowable number of failures since any further reduction causes the maximum risk values specified to be exceeded. Therefore, only test plans with r*>5 satisfy the given requirement. The graphs clearly show the existing trend: an increase in r* results in decreased risks but is also accompanied by longer test times.

2. Plans having unequal risks and separate constraints on each risk.

Example 2: Devise a test with α and β having specified but unequal values. For this example assume that an α of 18% and a β of approximately 13% is desired.

The procedure for obtaining the required plan is only slightly more complicated than that of the previous example. Again, it should be observed that establishment of viable test conditions requires that R*A equals R*B equals r*, and that both the allowable number of failures and test time have to be single valued. Within these restrictions, the needed T/θ_o and r* values are obtained by drawing a horizontal line through the α=18 point on the Y axis. The line should intersect all R*A curves. Pick an R*A curve, say R*A=7, and draw a vertical line from that point of intersection to the corresponding R*B curve (R*B=7, in this instance). The latter point of intersection provides the β value for this test, which is approximately 16%; when the vertical line is extended to the abscissa the corresponding test time, approximately 5.4 θ_o hours, can be obtained. Since the consumer's risk (16%) exceeds the desired value (13%) the procedure is repeated, this time choosing the R*A=8 curve. An α of 18% on this curve delineates a β of 12.0% and a corresponding test time of 6.3 θ_o hours. Since requirements are met, the latter plan is acceptable. In instances where some flexibility exists in the statement of test requirements, the graph can be utilized in other ways. For example, the data indicates that a 3% reduction in β can be realized by allowing test time to increase by a normalized unit (T/θ_o=1). Whether this represents an acceptable tradeoff depends on the constraints of the specific problem addressed. The example, however, illustrates another potential area of application for this procedure.

3. Plans having a time constraint and (possibly) a single risk constraint.

Example 3: Develop test plans where test length does not exceed 5.0 θ_o hours.

All feasible plans are delineated by drawing a vertical line through the T/θ_o=5.0 point and are situated on or to the left of that line.

Choosing the R*A=10 curve, an α of approximately 1.4% is obtained. However, the corresponding β cannot be obtained directly from the graph since its value exceeds 30%. By letting α take on larger values the corresponding β's become smaller. For α equal to 13% and using the previous test time of $T/\theta_o=5.0$, the appropriate β is 22%. This value is obtained from the point of intersection of the $T/\theta_o=5$ line and the R*B=7 curve. The selection of the R*B=7 curve is based on the requirement that R*B must equal R*A. (R*A=7 had previously been defined by choice of the α and T/θ_o values.) Other solutions can be obtained by choosing another feasible test time and following a routine similar to that described.

At this point it would perhaps be instructive to show how readily the test conditions for some fixed time plans of MILSTD 781B can be reproduced using the graphical procedure. For example, test plan XIV, in table 4, page 13, specifies a producer's risk of 10%, β=20%, requires 6.2 hours of normalized test time and allows [illegible] failures; test plan XV in the same table is a plan having an α=β=20%; a $T/\theta_o=3.0$ hours and allows 5 failures for acceptance. With the aid of figure 1 and use of any 2 of the 4 quantities specified for each plan, the other values can be readily verified. The reason for this particular choice of MILSTD 781B plans is that the values of the discrimination ratio, K, and the allowable number of failures, r*, are identical to those used in constructing this graph. However, all plans shown in that table can be derived graphically if the curves are redrawn incorporating the appropriate parameter adjustments.

C. Application to Bayes Test Plans

Test plans utilizing either a combination of Classical and Bayes (Hybrid) or both Bayes criteria cannot be formulated without some prior distribution. For reasons of convenience and because of its wide usage, the inverted gamma distribution* served as the prior distribution for all Bayesian and Hybrid plans described here. This density function can be expressed as:

$$f(\theta) = \left(\frac{\delta}{\theta}\right)^{\phi} \frac{e^{-\left(\frac{\delta}{\theta}\right)}}{\theta\Gamma(\phi)} \qquad \theta \geq 0$$

$$= 0 \qquad \theta < 0$$

where φ, δ denote the shape and scale parameter of the distribution. The mean of this distribution, $E(\theta)$, equals $\delta/(\phi-1)$ for $\phi>1$ and $\Gamma(\phi)$ is the gamma function of φ.

* Results from a change of variables such that the new variable is the inverse of the gamma distributed random variable.

The table shown on page III-6 summarizes the characteristics of the Bayes functions graphed and establishes a frame of reference for this discussion.

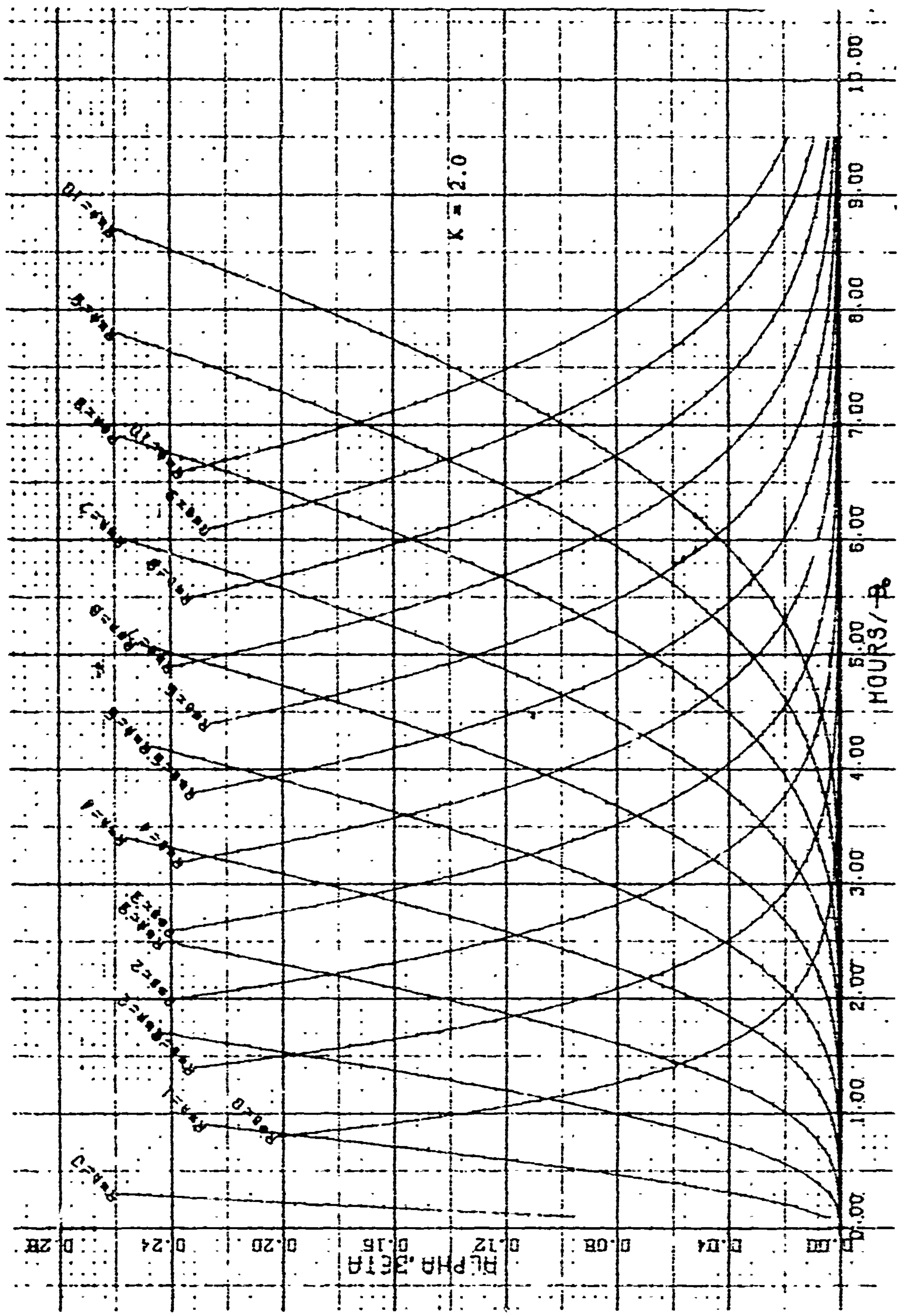

FIGURE 1 - Classical Method

Criteria Set	Parameter Values shape parameter	scale parameter	Expected Value $E(\theta)$	Discrimination ratio K	graph is shown in figure
A-B	3	150	75	2	2
$A-B_1$	3	150	75	2	3
A_1-B_1	3	150	75	2	4
P(Acc)-B	3	200	100	2	5

The shape of the Bayes risk functions, as can be observed from the graphs, resembles the Classical α, β curves, especially for the A_1-B_1 criteria set. Since the Classical and the A_1-B_1 risks have similar definitions, their likeness is not too surprising. The Bayes risks, however, are expressed in terms of a probability which is averaged with respect to a region of the prior distribution (i.e., the conditional probability of rejection is defined for values of $\theta \geq \theta_0$) whereas the classical risks are specified in terms of point values for θ.

This similarity also carries over, to some extent, to the other Bayes plans. A point of significant difference is that the Classical consumer's risk curve will always start at one (i.e. have a value of one for zero test time) and will asymptotically approach zero for large test time. The Bayes consumer's risk will generally not do this. The initial value is determined by the area in the prior density bounded by $0 \leq \theta \leq \theta_1$. Alternatively, the Bayes producer's risk curves will generally not converge to an asymtote of 1 for large test time as is the case with the Classical producer's risk. The limiting value reached depends on the area segment in the prior density defined by the limits $\theta_0 \leq \theta \leq \infty$. These features and their effect on test plan characteristics are examined in more detail in the next section.

Graphs of criteria set P(Acc)-B, shown in figure 5, exhibit the greatest difference compared to Classical plans. The usual approach to test design using this criteria set is to initially select values for P(Acc) and the consumer's risk and then calculate the other parameters. Since P(Acc) indicates the probability of successfully passing the test, it is in the producer's interest to specify a large value for this quantity, (i.e. within the limiting value of one), Of course the P(Acc) value should not be picked arbitrarily, but based on the prior parameter estimates, choice of allowable number of failures, r*, and test time, T. Increasing r* increases P(Acc) but it also increases T, for a given level of consumer's risk. Since it is generally desirable to keep T small, an inflated P(Acc) estimate is likely to reflect an expected test advantage. Since $E(\theta)$ is computed from the same parameter estimates it will also

be affected; the direction of change is such that an increase in P(Acc) will also increase E(θ). Test plans with large P(Acc) and E(θ) generally lead to very short tests as is evidenced by some plans appearing in reference (6). The problem with short tests is that the observed data generally is inadequate to influence the test decision.

The use of these plans is therefore not recommended. P(Acc) can, however, be constructively used in another manner; it can be employed as a technique for predicting the reliability status of a product in the various stages of its development and thus help to flag problems that may require corrective action.

The application of the graphical procedure to formulate Bayes tests generally follows along the lines previously discussed. However, several additional examples, with arbitrarily chosen parameters, will be presented to further illustrate this technique.

Example (4): Assume a Bayesian test, using criteria A-B, is to be developed with the requirement that it not exceed 2.0 θ_o test units. Determine all pertinent test parameters.

A vertical line drawn through the $T/\theta_o=2$ point on the abscissa delineates, to the left of that line, all practical test possibilities. Choice of $r^*=1$ for the allowable number of failures leads to a test plan with a producers risk, A, of approximately 9.5%; a consumers risk, B, of approximately 6% for $T/\theta_o=2$. Reducing test time to $T/\theta_o=1.75$ results in a plan where A=B=6% when $r^*=1$. Further reduction in test time to $T/\theta_o=1$ results in a test condition where A=B=12% and $r^*=0$.

The requirement that T/θ_o not exceed 2 units is met by all plans. The final selection process therefore must include other considerations. However, the ease with which viable alternatives can be identified through use of this procedure is worth noting.

Example (5): Assume that a given test problem does not dictate use of a specific Bayes plan and that all feasible Bayes tests will be considered for test plan selection. Specifically what is desired is a plan having equal risks, not to exceed 10%, and minimum test time.

The following plans using the A-B criteria depicted in figure 3 will meet the requirements:

a) $T/\theta_o=1.75$ $r^*=1$; A=B=8.6%
b) $T/\theta_o=2.5$ $r^*=2$; A=B=6.5%

There are several other possibilities but they are not practical because of excessive test lengths.

Some viable candidates using the A_1-B_1 criteria set, shown in figure 4 are:

a) $T/\theta_o=1.75$ $r^*=1$; $A_1=B_1=13.6\%$
b) $T/\theta_o=2.5$ $r^*=3$; $A_1=B_1=10.5\%$

The feasible plans of the A-B_1 criteria set, shown in figure 3, are:

a) $T/\theta_0=1.5$ $r^*=1$; $A=B_1=7.8\%$
b) $T/\theta_0=2.25$ $r^*=2$; $A=B_1=6\%$

A comparison of plans with approximately equal test time (i.e., the comparison is among plans (a) or plans (b) in the different test categories) indicates plans using the A-B or the A-B_1 criteria are preferable to the A_1-B_1 plans, since they offer lower risks. It is to be emphasized, however, that the purpose of this example is to illustrate use of a technique; not to suggest that different test criteria can summarily be subjected to a trade-off study; or to imply that selection of a preferable plan be based solely on minimum risk or minimum test time. A course of action for these situations is proposed in a subsequent section of this report.

Example (6): Devise a Bayesian test plan which has an a priori probability of acceptance, P(Acc), of 90%.

Draw a horizontal line through the P(Acc)=.9 point on the Y Axis of the curves shown in figure 5. The intersection of this line with each r* curve (the r*=0 curve is not used because the point of intersection is too close to the time origin) identifies potential candidates in terms of test time, P(Acc) and r*. To obtain the corresponding value of consumer's risk construct a vertical line from a given point of intersection to the consumer's risk curve having the same r* value. Thus, for the r*=1 curve, a B value of .21 and a test time of .3 θ_0 units is obtained. For the r*=2 curve, the corresponding value is B=20, T=.7 θ_0 units; the r*=6 curve yields a B of .16 and T=2.1 θ_0 units. The relatively short test times indicated for these plans, even for the r*=6 plan, tends to confirm previous remarks concerning the expected shortcomings of these tests.

D. Hybrid Test Plans

The outstanding feature of Hybrid plans is that the consumer's risk is not affected by the prior distribution. This property makes utilization of these plans especially attractive in cases where difficulty is experienced in the assignment of an appropriate prior or when assumptions underlying its formulation are tenuous. As a test method Hybrid plans offer a number of advantages over the Classical approach, some of which will be discussed in the next section. However when viewed in terms of their consumer/producer risk curves they offer nothing new. The Classical B is the consumer's risk function used in all the graphs depicting this test method, while several different Bayes criteria are used for the producer's risk. Figure 6 denotes the A-B combination; figure 7 the A_1-B, and figure 8, the P(Acc)-B set. A priori parameter values are identical to those previously used, i.e., $\delta=150$, $\phi=3$.

In addition to the obvious application as a device for test plan development, the graphs have utility in other respects as well. For example, when evaluating alternative approaches it may be of interest

to determine the value of the Classical consumer's risk that corresponds to the consumer's risk of a given Bayes plan. To illustrate how this may be accomplished, the three different Bayes plans of example 6, with the following properties, are utilized;

1) $T/\theta_0=2.0$; $r^*=1$; $A=9.5\%$; $B=6\%$
2) $T/\theta_0=2.75$; $r^*=1$; $A=B=8.6\%$
3) $T/\theta_0=1.0$; $r^*=0$; $A=B=12\%$

Example (7): The classical β's that conform to the β risks listed above can be ascertained with the aid of the graph of the Hybrid method contained in figure 6. (Conformance is established in the sense that this value of β determines a Hybrid plan which is identical to the original Bayes plan in all other test parameter values.) Any combination of two of the three quantities (T/θ_0, r^*, A) listed above can serve to locate the appropriate value. In the first plan, using $T/\theta_0=2.0$ and $r^*=1$, a β value of 9.5% is obtained. For the other plans, β's equal to 13.5% and 13.0% were obtained by following the same procedure.

Example (8): Perhaps another interesting example is one involving comparison of plans using different criteria. Specifically what is wanted is a measure of the differences between plans with A-β and A_1-β criteria sets.

Maintaining the consumer's risks fixed at a value of 10%, one possible plan, using risks A_1-β has a producer's risk of 21% for $T/\theta_0=3.3$ and $r^*=3$. Another plan using the A-β criteria, again with consumer's risk of 10%, has a producer's risk of 12.4% and requires only $T/\theta_0=1.15$ test units for $r^*=0$. If the test lengths are made equal by adjusting the time of the A-β plan, this plan will now have a producer's risk of approximately 5.8% while the consumer's risk remains at 10%.

An equal risk plan using criteria A_1-β requires $T/\theta_0=5.25$ units of time allows six failures and has risks of $A_1=\beta=10.4\%$. For criteria combination A-β, a comparable plan in terms of the risks, has $A=\beta=9.5\%$ and can be performed in 2 θ_0 test hours, allowing one failure. The difference in test time, while substantial here, depends on the values assigned to the a priori parameters. Again, these examples are not intended to imply that one set of criteria is readily interchangeable with other sets; rather it is suggested that a careful review of candidate criteria sets, to determine whether they represent meaningful figures of merit in a given problem situation, precede a comparison of the type outlined above.

In the previous section it was recommended that P(Acc) not be used as a test criterion but as a means of deciding whether the device to be tested is ready for test or in need of rework. The graph shown in figure 8 can be used to calculate the value of P(Acc) on which this decision can be based. To implement this requires that any two of the four quantities that define the test (i.e.; T/θ_0; r^*, consumer's risk, producer's risk) be available in numerical form; the appropriate P(Acc) value can then be determined from the graph of figure 8.

SECTION IV

I. CHANGES IN TEST PLANS AS A FUNCTION OF CHANGES IN PARAMETER VALUES.

A. General

The aspect of Bayesian tests that is of much practical concern is the influence of the prior distribution on the test. Since it generally is difficult to arrive at reliable estimates for the prior parameters, it is of considerable interest to determine, for example, how errors in these estimates change preselected risks or otherwise affect test properties. Although there are a number of articles in the literature that attempt to deal with this problem, conclusions differ and seem to depend on the variables analyzed.

For the inverted gamma prior distribution, this study has found that the Bayes risks are quite sensitive to the expected value of this distribution, E (θ). In fact the difference between $E(\theta)$ and the required MTBF, θ_0, constitutes a useful easure in the analysis of this problem. It should be recalled that $E(\theta)$ represents the value of MTBF derived, either empirically or on a personal probability basis, prior to conducting the test. θ_0, on the other hand, is the value of MTBF that reflects the requirements of the intended application. The demonstration statement is usually posed in terms of another value of MTBF, θ_1, denoting a minimum acceptable MTBF. When $E(\theta)$ is large compared to θ_0, for a given θ_1, there is favorable expectation of successfully completing the test. This condition is designated as the optimistic case, or one with an optimistic prior, and is in contrast with the pessimistic case, where $E(\theta)<\theta_0$. The latter case reflects a situation where the pre-test MTBF estimate is less than the value specified for the test. Consequently, the chance of successfully passing the test is small, assuming the estimate is reasonably representative. In the optimistic case, a large area segment in the prior distribution is generated by values of θ between the limits $\theta_0 \leq \theta \leq \infty$; whereas in the pessimistic case, values of θ within the range $0 \leq \theta \leq \theta_1$, will give rise to a large area. That is, large left or right tail areas result when the product is estimated to be either very bad or very good compared to the required value, θ_0.

The approach used to study the impact of the prior consisted of constructing graphs of selected Bayes and Hybrid risk functions for various values of expected MTBF, $E(\theta)$, and to make inferences concerning the differences observed. Specifically, criteria sets A-B, A-ß, A_1-B_1 and A_1-ß were selected for this analysis. Five graphs were plotted per criteria set, each with a different value of the scale parameter, δ, as follows: $\delta \equiv$ 100, 150, 200, 300, 400. As a consequence, $E(\theta)$ took on values of 50, 75, 100, 150 and 200, respectively. θ_0 was chosen to equal 100, and θ_1 was 50. This assignment permitted examination of both optimistic and pessimistic priors.

B. Effect on Bayes Plans

Figures 9-13 are plots of the risk functions of criteria set A-B vs time. In the initial graph $E(\theta)$ equals 50 and since $\theta_0=100$, it exemplifies the pessimistic case. In the succeeding graphs $E(\theta)$ takes on increasingly more optimistic values until a value of 200 is reached as shown in Figure 13. The different initial and final values of the risk curves, a condition previously noted, are readily observable in these graphs. As explained, these values represent probabilities which are derived from the prior density; moreover, their relative magnitudes can be inferred directly from the classification assigned to the prior (i.e., a prior is classified as pessimistic or optimistic relative to the test requirement, θ_0. Also, a pessimistic prior implies a large left tail area; an optimistic prior a large right tail area.)

The relationship between the size of the tail areas and the limiting values assumed by the risk functions follows directly from the definition of these risks. For example, by definition, the Bayes producer risk is a conditional probability indicating the fraction of rejected units that are good (good means that $\theta\geq\theta_0$). For long tests, this conditional probability will equal $P(\theta\geq\theta_0)$, the probability of having good units on test. However, $P(\theta\geq\theta_0)$ is also the right tail area of the prior distribution. Therefore, an optimistic prior will give rise to a large asymptotic value of producer's risk. Conversely, a small limiting value of producer's risk is caused by a pessimistic prior. Moreover, as this function increases monotonically with time the limiting value also represents the maximum value.

Similarly, the consumer's risk which is defined in terms of the percentage of accepted items that are bad (i.e., have a $\theta\leq\theta_1$) will, as test time approaches zero, equal the percentage of bad items on test, $P(\theta\leq\theta_1)$. For a pessimistic prior this will be a large value. As more optimistic cases are encountered, by increasing $E(\theta)$, this magnitude will decrease. Finally, when the expected MTBF is twice the required value (i.e., $E(\theta)=2\theta_0$), the situation depicted in Figure 13, the consumer's risk curves will not only have a small initial value but will be essentially independent of test time over an extended portion of total test time. Since these curves decrease monotonically with time, the initial value also represents the maximum value. Thus, with an optimistic prior it is possible to formulate test plans having, both, a very small consumer's risk and a very short test time, as exemplified by plans that may be developed from the graph of Figure 13. The basic problem with this type of test is its almost total dependence on the prior MTBF assignment. When it can be established that the prior does accurately reflect the MTBF characteristic of concern, these plans may be attractive in terms of the cost savings they offer. Difficulties arise, however, when this is not the case. The tabulated data shown below indicates how rapidly the risks change as a function of the prior parameter values. The data was derived from the curves of Figures 9-13 using a normalized test time $T/\theta_0=1.5$ units and $r^* = 1$ for all data points shown.

$E(\theta)$	δ	A	B
50	100	.028	.22
75	150	.078	.114
100	200	.15	.06
150	300	.3 (estimated)	.014
200	400	(cannot be determined from the graph)	.002

As indicated in this table, a 4 to 1 variation in $E(\theta)$, or δ_1, causes approximately a 100 to 1 change in the consumer's risk and a correspondingly large change in the producer's risk. While the selection of the T/θ_0, r^* values used in this example may seem arbitrary, the results are indicative of the magnitude of the risk excursions that will generally be encountered. This may be verified, with the aid of the graphs, in the following manner: Select an arbitrary set of risk curves (i.e., one consumers risk curve and a producers risk curve with the same r^* value) and examine the change in the risk functions as $E(\theta)$ is varied. This is best accomplished by picking a number of points on the time axis and determining the risk values corresponding to these points for each graph shown in Figures 9-13. Review of this data will permit determination that, for a given r^*, as T/θ_0 is increased the variation in $E(\theta)$ with consumer's risk will decrease, while the opposite is true for the producer's risk. However, since short tests have greater practicality and are therefore utilized more frequently, the previous conclusion is generally applicable.

Bayes plans utilizing criteria set A_1-B_1 are shown in Figures 14 to 18. The characteristics of this set were not examined in detail although it was noted that changes in parameter values have a much smaller effect on the risk functions and that the different initial and final values previously observed were absent, at least over the range of values plotted in the graphs.

C. Effect on Hybrid Plans

Figures 19-28 illustrate the impact of parameter changes on Hybrid plans. As can be seen, the consumer's risk function is not affected by these changes while the variation in the producer's risk curves is identical to those displayed in previously presented graphs, appearing in Figures 9-13. The fact that consumer's risk is independent of the prior distribution precludes implementation of plans with, both, a small consumer's risk and a short test time.

In addition to providing good consumer protection, Hybrid tests offer another advantage in that they generally can be completed within a reasonable time frame. However, a quantitative comparison between Classical and Hybrid plans, necessitates establishment of a specific set of ground rules. Based largely on the thought that a Bayes and a classical producer's risk are alike in the sense that both represent decision errors whose magnitude is to be controlled by the test, an equal value assignment was made. Equal values were also assumed for the consumer's risks, because of the resulting simplification in the data. The pertinent data may be arranged as shown below:

Choice of α = A = 13% and β = 13% provides the following plans:

K=2

θ_0=100	E(θ)	r*	T/θ_0	β	A	B	Comment
Classical	-	10	7.4	13%	-	13%	
Hybrid	50						No plans are possible for these risks
	75	0	1.0	-	12	12	
	100	2	2.5	-	12.5	12.5	
	200	8	6.5	-	13	13	These are extrapolated results; r* values have to be increased beyond the number shown in the graph to get definitive points.

Choice of A = α = β = 5.5% results in the following plans:

K=2

θ_0=100	E(θ)	r*	T/θ_0	β	A	B	Comment
Classical	-	14	10	5.5	-	5.5	Extrapolated value
Hybrid	50	0	1.5	-	5.5	5.5	
	75	4	4.5	-	5.5	5.5	
	100	7	6	-	5.5	5.5	Extrapolated value; r* must be greater than 6 to get specific test points.
	150			-			
	200						

The T/θ_0 column indicates the amount of test time required for each plan.

The tabulation shows that Hybrid plans require less test time than their classical equivalents (equivalency is in terms of the risk values), with the largest savings in test time occurring for Hybrid plans with pessimistic priors.

D. Effect on the Discrimination Ratio.

To this point, a discrimination ratio, K of two has been used in all the graphs presented. The fact that test properties change as the K ratio is changed is a well documented result for the Classical method and applies to the Bayesian and Hybrid methods as well. Increasing the K ratio decreases test time for a given level of risks and allowable number of failures. However, the new operating characteristic (O.C.) curve adversely affects the consumer if the change in the K ratio is accomplished by decreasing θ_1. If θ_0 is modified the producer is penalized. In this report, the different K ratios were obtained by changing θ_1, analogous to the procedure used in MIL STD 781-B.

Although all test methods exhibit this sensitivity, Bayes plans utilizing criteria set A-B were chosen to illustrate this effect. Two additional K values, K=1.5 and K=3, and two values of $E(\theta)$, $E(\theta)=50$ and $E(\theta)=100$, were utilized in this analysis. The pertinent graphs are shown in Figures 29-34.

It is to be observed that there is no change in the producer's risk curves as a function of the different K values. That is, the family of producer's risk curves for $E(\theta)=50$ and $E(\theta)=100$ are identical for K=1.5, 2 and 3. This is explained by recalling that the change in the discrimination ratio results from a change in θ_1. Since the producer's risk does not involve θ_1, this function is not affected. However, the consumer's risk function, which does contain θ_1 is noticeably different as the K ratio is varied. As in the Classical Case, the change is in a direction which requires less test time when the K value is increased, if the other parameters are held constant. The graphs for $E(\theta)=50$, Figures 29-31, indicate, for example, that a test plan with risks A=B=5% and r*=0 requires a T equal to one θ_0 unit when the K ratio has a value of 3; when this ratio is decreased to 1.5, test time increases to 5.5 θ_0 units while the other parameters are kept at approximately the same values (i.e., A=B=4.5% and r*=0).

The consumer's risk curves for $E(\theta)=100$, Figures 31-34, in addition to the previously noted effects, exhibit different initial values when the K ratio is varied (i.e., have different values of consumer's risk for test time approaching zero). This type of behavior has been observed previously but under different conditions. Earlier, when discussing this result in connection with a Bayes test, θ_0 and θ_1 remained fixed as the prior parameters and, therefore, $E(\theta)$ took on different values.

In the present situation, $E(\theta)$ and θ_0 are constant, while θ_1 and consequently, K, is varied. Clearly this is a different condition; yet one which is amenable to a similar explanation providing the original context is widened. The following rules apply to the more general case:

a. For constant $E(\theta)$

An increase in θ_0, θ_1 or both, creates a more pessimistic prior (i.e., it decreases the chance of successfully passing the test); a decrease is θ_0, θ_1 or both, produces a more optimistic prior.

b. For constant θ_0, θ_1

An increase in $E(\theta)$ leads to a more optimistic prior; a decrease in $E(\theta)$ makes the prior more pessimistic.

The new framework helps to emphasize that the prior estimate as well as the test specification determine the classification of the prior distribution. The graphs of Figures 13 and 34 tend to underscore this fact. As can be observed, the initial values of the consumer's risk functions are similar even though the $E(\theta)$ and θ_1 values differ considerably. The explanation is that, in accordance with the stated rules, both priors are to be classified as optimistic priors. This is the reason for the similarity in the initial values of the respective risk functions observable on the graphs.

SECTION V

I. TEST PLAN SELECTION CRITERIA

There are two fundamental aspects to reliability demonstration, economic and statistical. Because of severe resource constraints, current practice is to choose a test based primarily on economic considerations. However, statistical issues cannot be ignored as they affect the essential purpose and usefulness of this type of test.

The basic aim of demonstration testing is to distinguish between two possible values in a quality characteristic of a product, such as MTBF. For the Classical method, the instrument that measures how well a test performs this function is called the Operating Characteristic (O.C.) curve of the test. The curve provides a quantitative method for judging proposed plans and has long served as a criterion for test plan selection. Unfortunately, the Bayes approach does not make use of a similar procedure. The function that is often calculated is the probability of acceptance, P(Acc), a quantity that has already been discussed. P(Acc) is the classical O.C. curve averaged with respect to the entire prior distribution which, for a given T, r* combination, takes on a single value. The use of this numeric as a figure of merit has the disadvantage in that it includes all the uncertainties inherent in the prior distribution. Consequently, information imparted in this way tends to be qualitative and more difficult to interpret compared to what is available using the conventional approach. Moreover, as test specifications, for all methods described, are stated in terms of specific values of MTBF and not in terms of distributions or averages, it seems appropriate to continue use of O.C. curves as a tool for the evaluation and comparison of any test plan, no matter how derived.

One approach to determine the O.C. curve of a test plan is to substitute the T, r* values of that plan into the equation for P(Acc/θ), (i.e., the conditional probability of acceptance for a given value of θ), postulate different values for θ and compute P(Acc/θ) for these values. Since a constant hazard rate model is assumed by all test methods, the appropriate expression for P(Acc/θ) is the Poisson function summed over the allowable number of failures. Stated in equational form,

$$P(Acc/\theta) = \sum_{r=o}^{r^*} \frac{\left(\frac{T}{\theta}\right)^r e^{-\left(\frac{T}{\theta}\right)}}{r!}$$

where T denotes test length; r equals the number of failures occurring in T; r* is the maximum number of failures allowed for acceptance and T/θ is the average rate of failure occurrences. Thus, once the pair (T, r*) of a plan is known, its O.C. curve can be calculated. Also, since a T, r* combination determines a unique test plan, its O.C. curve is also unique. This approach, therefore, provides a logical and objective basis for judging how well expected test goals are being achieved. In summary, the suggested procedure offers the following advantages:

a. It is consistent with the form in which the test hypothesis is posed: i.e., in terms of single MTBF values, not in terms of distributions or averages. The Classical O.C. curve is well suited to provide a concise answer to a question that normally arises in this context, which is: how does the proposed test perform if the product has an MTBF equal to the value(s) postulated?

b. It is not directly dependent on the prior distributional assumptions. The prior, however, partially determines the T,r* values which are used to compute the O.C. curve.

c. It provides a common basis for comparing different plans no matter how derived.

d. It is simple to construct and easy to evaluate.

e. It permits examination of a plan in terms of an attribute whose meaning is unambiguous and well understood.

Given that T, r* values have been chosen, the actual calculation of the O.C. curve can be accomplished in several ways. Since the solution involves use of Poisson probabilities, which are widely tabulated or available in chart format (Thorndyke chart), a feasible method is to reference existing aids and provide instructions on their use. This approach, however, introduces some inconvenience in performing the necessary computations and was, therefore, not implemented. Instead, it was decided to develop a simple graphic scheme capable of addressing this problem more directly; it consists of plotting contours of constant normalized MTBF values, θ/θ_0, in the probability of acceptance P(Acc) and normalized test time, T/θ_0, plane. Each graph contains a family of 20 θ/θ_0 curves which vary between the limits of .1 and 2, in increments of .1. Also, a separate graph is constructed for each value assigned to r*. Ten values of r* were selected, starting with zero and increasing in unit steps to r*=9. With the aid of these graphs, the O.C. curve of any plan can be determined by means of a simple two step procedure:

1. Given the T, r* values of a plan, select the graph with the same r* number as that specified by the plan; (for example, if the test plan specifies r*=6, choose Figure 41).

NOTE: Two different symbols, R* in the graphs and r* in the report, have been used to deonte the same quantity. .1. R*=r*

2. Having identified the proper graph, draw a vertical line, from the point on the abscissa equal to the T value of the test, to intersect all 20 curves of the graph. Each point of intersection identifies a value for the acceptance probability which can be read on the ordinate scale. The θ/θ_0 value associated with a particular point of intersection can be ascertained as follows: The first θ/θ_0 curve, i.e. the one closest to the horizontal axis, has a value of θ/θ_0=.1; the

next curve has a value of .2, etc. Thus, an O.C. curve can be constructed based entirely on the data derived from the 20 points of intersection. The following example will illustrate this procedure in more detail.

NOTE: The graphs are scaled in terms of normalized test units. Therefore, if the plan measures time in T/θ_0 units, this value should be entered directly, otherwise the specified test time must be divided by θ_0.

Example (8). Example 2 of this report, discusses a Classical test plan with the following characteristics: $\alpha = 18\%$; $\beta = 12\%$; $T/\theta_0 = 6.3$; $K = 2$, $r^* = 8$. Determine the O.C. curve for this plan.

Since $r^* = 8$, the proper graph is Figure 43. Locate the $T/\theta_0 = 6.3$ point on the abscissa of that graph. Use a ruler to draw a vertical line from this point to intersect all curves in that graph. A horizontal line drawn from each point of intersection to the Y axis, provides the corresponding values of the probability of acceptance.

The data is most suitably arranged in terms of the tabulation shown below:

θ/θ_0	P(Acc) percent
.1	0
.2	0
.3	0
.4	2.5
.5	12
.6	28
.7	48.5
.8	61
.9	73
1.0	81.5
1.1	86
1.2	91.5
1.3	14
1.4	96
1.5	97
1.6	98
1.7	98.5
1.8	99
1.9	99.2
2.0	99.5

The O.C. curve of this test can be constructed entirely from the above data. However, salient test properties can be inferred without actually doing this. First note that for $\theta/\theta_0 = 1.0$, the probability of acceptance equals 81.5%. Since the test was constructed to have an α of approximately 18%, the O.C. data shows that this requirement is approximately satisfied (i.e., since $P(Acc/\theta=\theta_0) = 1-\alpha$). Similarly,

when $\theta/\theta_0 = .5$, the corresponding P(Acc) value is 12%. This correctly reflects another requirement of this plan, which is that β=12%=P(Acc/θ=θ_1), when the discrimination ratio is 2. Going beyond these two points, the protection afforded by this test can also be ascertained without difficulty. The region of main concern to a consumer encompasses acceptance probabilities for small values of θ. If these probabilities are small, this indicates that he is getting good protection from the plan. For example, a device with an MTBF of 40% of the required value will only have a 2.5% chance of being accepted by the test. Conversely, the portion of the O.C. curve of primary interest to a producer concern the acceptance probabilities for large values of θ. The data indicates that a device having an MTBF that is 120% of the required value has a 91.5% chance of being accepted. This O.C. curve, therefore, reflects the condition that, in general, good devices will be passed and poor devices rejected by the test, consonant with the basic purpose of this type of test.

Having illustrated this procedure on a Classical Plan, it is informative to apply the technique to other test methods as well. Since the procedure is the same no matter what plan is used, the intermediate steps are omitted and only final results are shown. Extending the analysis of a previous example (example 5, page III-7), the O.C. data for each plan cited in example 5 was calculated and is presented in the following table.

O.C. Tabulations for Test Plans of Example 5

NOTE: The common features of these test plans are: K=2, δ=150; ϕ=3; E(θ)=75; θ_0=100. Characteristics that vary from plan to plan are indicated in the data column of that plan.

	A-B		A_1-B_1		A-B_1	
θ/θ_0	A=B=8.6% r*=1 T/θ_0=1.75	A=B=5.5% r*=2 T/θ_0=2.1	A_1=B_1=10.5% r*=3 T/θ_0=2.45	A_1=B_1=8.5% r*=4 T/θ_0=3.15	A=B_1=7.8% r*=1 T/θ_0=1.5	A=B_1=6% r*=2 T/θ_0=2.25
.1	0	0	0	0	0	0
.2	0	0	0	0	0	0
.3	2	3	4	2	4	2
.4	7	10	14	10.5	11	7.8
.5	13.5	21	28.0	22	20	17
.6	21	32	42	40	29	27.5
.7	29	42.5	53.5	53	37	37.5
.8	36	51	63.5	64	44	47
.9	42	59	71	72.5	50	55
1.0	47.5	65	77	79	56	61
1.1	52.5	70	81.5	84	60	66.5
1.2	57.5	74.5	85	87.5	65	71
1.3	61	78	88	90	68	75
1.4	64	81	90	92	71	78
1.5	67.5	83.5	92	94	73.5	81
1.6	70	85	93	95	76	83
1.7	72.5	87	94	96	78	85
1.8	75	88.5	95	96.5	80	87
1.9	76.5	90	96	97	81.5	88.5
2.0	78	91	96.5	97.5	83	90

By way of comparison, the data indicates that the protection provided by these plans is inferior to that of the Classical plan of the previous example. This result is not completely unexpected since the test time for the latter plan is considerably greater than the times required by the Bayes test. Comparisons within Bayes plans identify plan A-B as being best timewise, but lacking in protection to either consumer or producer. Plan A_1-B_1 does better for the producer and a little worse for the consumer. The poor producer protection exhibited by these plans, especially plan A-B, is somewhat surprising especially in view of the tendency for contractors to favor the Bayesian approach. However, this data cannot be considered representative since it is based on arbitrarily selected prior parameter values. To address the more general condition, the O.C. characteristics of four Bayes plans of criteria set A-B, each having a different prior, were calculated and this data appears in the following table. All pertinent test conditions are as shown in the table. To facilitate comparisons, an attempt was made to keep the risks approximately the same.

O.C. Characteristics for Bayes plans using criteria set A-B

NOTE: Common features of the plans are K-2; θ_0=100. Other conditions are as indicated in the table.

θ/θ_0	A=B=6% E(θ)=50 δ=100 ϕ=3 T/θ_0=2 r*=0	A=B=8.6% E(θ)=75 δ=150 ϕ=3 T/θ_0=1.75 r*=1	A=B=8.6% E(θ)=100 δ=200 ϕ=3 T/θ_0=1.6 r*=2	A=B=5% E(θ)=150 δ=300 ϕ=3 T/θ_0=1.25 r*=4	A=B=1.4% E(θ)=200 δ=400 ϕ=3 T/θ_0=.4 r*=6
.1	0	0	0	0	87.5
.2	0	0	1	25	99.5
.3	0	2	9.5	60	100.0
.4	0	7	23.5	80	
.5	2	13.5	38	89	
.6	3.5	21	50	94	
.7	5.5	29	60	96.5	
.8	8	36	68	98	
.9	11	42	74	99	
1.0	13.5	47.5	78.5	99	
1.1	16	52.5	82	100.0	
1.2	19	57.5	85		
1.3	21.5	61	87.5		
1.4	24	64	89		
1.5	26	67.5	91		
1.6	28.5	70	92		
1.7	31	72.5	93		
1.8	33	75	94		
1.9	35	76.5	94.5		
2.0	37	78	95		

This tabulation illustrates a previously mentioned problem with some Bayes tests. It was pointed out on page II-5, that, if the prior is overly optimistic about the capabilities of a device, a Bayes procedure can be very poor in detecting inadequate devices. The O.C. data in the last column (i.e., where $E(\theta)=200$, twice the required value) confirms this statement. The data shows, for example, that a device having only 10% of the required MTBF value will have an 87.5 chance of being accepted by this test. The adjacent column, with a less optimistic prior, does better, but not by much. For $E(\theta)=150$, devices with an MTBF equal to 50% of the required value, will be accepted 89% of the time. This constitutes very poor consumer protection, although the producer does benefit in that almost nothing gets rejected. Resorting to plans with less optimistic priors helps the consumer, but this gain is accomplished at the expense of the producer's interest as his protection is drastically reduced. This is exemplified by the plan having an $E(\theta)$ of 75; that plan indicates that a device that is twice as good as the required MTBF has only a 78% chance of being accepted by the test. This, of course, represents an intolerable situation to a contractor. His interest is better served with a more optimistic prior but unfortunately, this causes consumer protection to deteriorate rapidly, as has been shown. In general, the use of an extreme prior results in a test which favors either producer or consumer, depending on which extreme is chosen.

The O.C. data for several additional tests previously discussed in this report, was compiled to permit more comprehensive evaluation of their properties. The next table shown is for plans using the A_1-B_1 criteria set. The plans also differ in respect to values assigned to the parameters of the prior distribution. It is to be noted that these plans provide better balanced protection than the A-B plans previously described.

O.C. Tabulation for test plans using criteria set A_1-B_1

Note: Common features of these plans are: $K=2$; $\theta_o=100$.

θ/θ_o	$A_1=B_1=9\%$ $\delta=100$ $\phi=3$ $T/\theta_o=2.25$ $r^*=3$	$A_1=B_1=8.5\%$ $\delta=150$ $\phi=3$ $T/\theta_o=3.15$ $r^*=4$	$A_1=B_1=9\%$ $\delta=200$ $\phi=3$ $T/\theta_o=3.3$ $r^*=4$	$A_1=B_1=9.2\%$ $\delta=300$ $\phi=3$ $T/\theta_o=3.5$ $r^*=4$	$A_1=B_1=8.6\%$ $\delta=400$ $\phi=3$ $T/\theta_o=3.7$ $r^*=4$
.1	0	0	0	0	0
.2	0	0	0	0	0
.3	5.5	2	1.5	1	0
.4	18.5	10.5	8	6	5
.5	34	25	21	17.5	14
.6	48	40	35	31	26
.7	60	53	49	44	39
.8	69	64	60	56	51
.9	75	72.5	69	65	61
1.0	81	79	76	72.5	68.5
1.1	85	83.5	81	78.5	75
1.2	88	87.5	85.5	83	80
1.3	90	90	88.5	86.5	84
1.4	92	92	91	89	97
1.5	93.5	94	92.5	91.5	90
1.6	94.5	95	94	93	91.5
1.7	95.5	96	95	94	93
1.8	96	96.5	96	95	94
1.9	97	97	96.5	96	95
2.0	97.5	97.5	97	97	96

The O.C. data for the important Hybrid test category has been assembled in several different ways. First, the data for the Hybrid plans of example 8, which involved a comparison of plans utilizing criteria sets A-β and A_1-β, is presented. In this tabulation, the 3rd column contains data of a comparable Classical plan. Test times, risks and other features of each plan are as indicated in this table. Next, the O.C. characteristic of each plan is depicted as a function of the prior characteristics and is presented in the subsequent two tables. The consumer protection provided by these plans is generally good (of course, at the β point, the risk has the specified value). The producer's interest is also well served in some of these plans especially when the prior is optimistic. The use of an optimistic prior in this test method does not lead to a breakdown in consumer protection as it does in some Bayes tests, but tends to be of mutual benefit.

O.C. Tabulations for test plans of Example 8

Note: the common features of these plans are: K=2; δ=150; ϕ=3; E(θ)=75; θ_o=100

θ/θ_o	A_1-β A_1=B=10.4% r*=6 T/θ_o=5.25	A-β A=B=9.5% r*=1 T/θ_o=2	α-β α=B=15.3% r*=8 T/θ_o=6
.1	0	0	0
.2	0	0	0
.3	0	1	0
.4	2.5	4	3.5
.5	10	9	15.5
.6	23	15.5	33.5
.7	37.5	22	51.5
.8	51.5	29	66
.9	63	35	77
1.0	72.5	40.5	84.5
1.1	79.5	46	90
1.2	85	50.5	93
1.3	88	54.5	95
1.4	91.5	58.5	97
1.5	93.5	61.5	97.5
1.6	95	64.5	98
1.7	96	67.5	98.5
1.8	97	70	99
1.9	97.5	72	99.5
2.0	98	74	99.6

O.C. characteristics of Hybrid plans using criteria set A-β

Note: common characteristics are: K=2; θ_o=100

θ/θ_o	E(θ)=50 A=β=5.4% δ=100 ϕ=3 T/θ_o=1.45 r*=0	E(θ)=75 A=β=5.6% δ=150 ϕ=3 T/θ_o=4.5 r*=4	E(θ)=100 A=β=6.9% δ=200 ϕ=3 T/θ_o=5.65 r*=6	E(θ)=150 A=β=13.4% δ=300 ϕ=3 T/θ_o=5 r*=6	E(θ)=200 A=β=21.4% δ=400 ϕ=3 T/θ_o=4.5 r*=6
.1	0	0	0	0	0
.2	0	0	0	0	0
.3	0	0	0	0	0
.4	2.5	1	1	3	7
.5	5.0	5	6.5	13	20.5
.6	8.5	8	17	27.5	37.[illegible]
.7	12.5	23	30	43	53.5
.8	16	34	44	56.5	66.5
.9	20	44	56	68	76
1.0	23	53.5	66	76	83
1.1	26.5	61	74	82.5	88
1.2	30	68	80	87	91.5
1.3	32.5	73.5	85	90	93
1.4	35	78	88.5	93	95
1.5	37.5	82.0	91	95	97
1.6	40	84.5	93	96	97.5
1.7	42.5	87	95	97	98
1.8	45	89	96	97.5	98.5
1.9	47	91	96.5	98	99
2.0	48	92.5	97	98.5	99.5

O.C. characteristics of Hybrid plans using criteria set A_1-β

Note: common characteristics are: K=2; θ_0=100

θ/θ_0	E(θ)=50 A_1=β=10.8% δ=100 ϕ=3 T/θ_0=5.2 r*=6	E(θ)=75 A_1=β=10.2% δ=150 ϕ=3 T/θ_0=5.25 r*=6	E(θ)=100 A_1=β=9.8% δ=200 ϕ=3 T/θ_0=5.3 r*=6	E(θ)=100 A_1=β=11.3% δ=200 ϕ=3 T/θ_0=4.5 r*=5	E(θ)=150 A_1=β=10.0% δ=300 ϕ=3 T/θ_0=4.65 r*=5	E(θ)=200 A_1=β=10.5% δ=400 ϕ=3 T/θ_0=3.95 r*=4
.1	0			0	0	0
.2	0			0	0	0
.3	0			0	0	0
.4	2.5			3	2.5	3
.5	10.5	10	9.5	11.5	10	10
.6	21.5			24	21.5	20.5
.7	39			35	35	32.5
.8	52.5			51	47.5	44
.9	64			62	58.5	52
1.0	73	72.5	72.0	70	67.5	63
1.1	80			77	75	70
1.2	85			82	80	76.5
1.3	88.5			86.5	85	80
1.4	91.5			90	88	84
1.5	93.5			91.5	90	87
1.6	95			93.5	92.5	89
1.7	96.5			95	94	91
1.8	97			96	95	92.5
1.9	97.5			97	96	94
2.0	98			97.5	96.5	95

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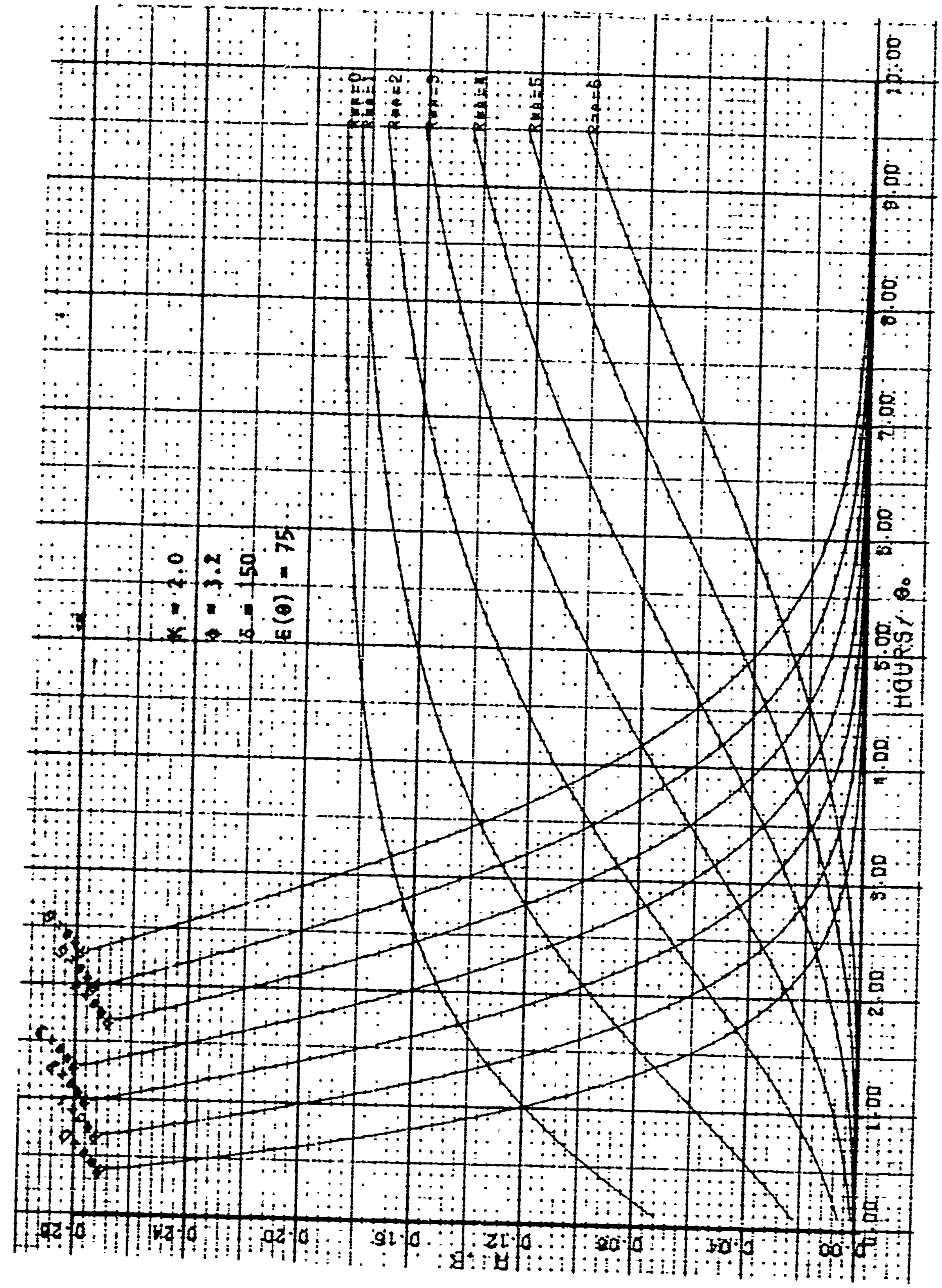

FIGURE 2 - Bayes Method; Criteria Set A-B

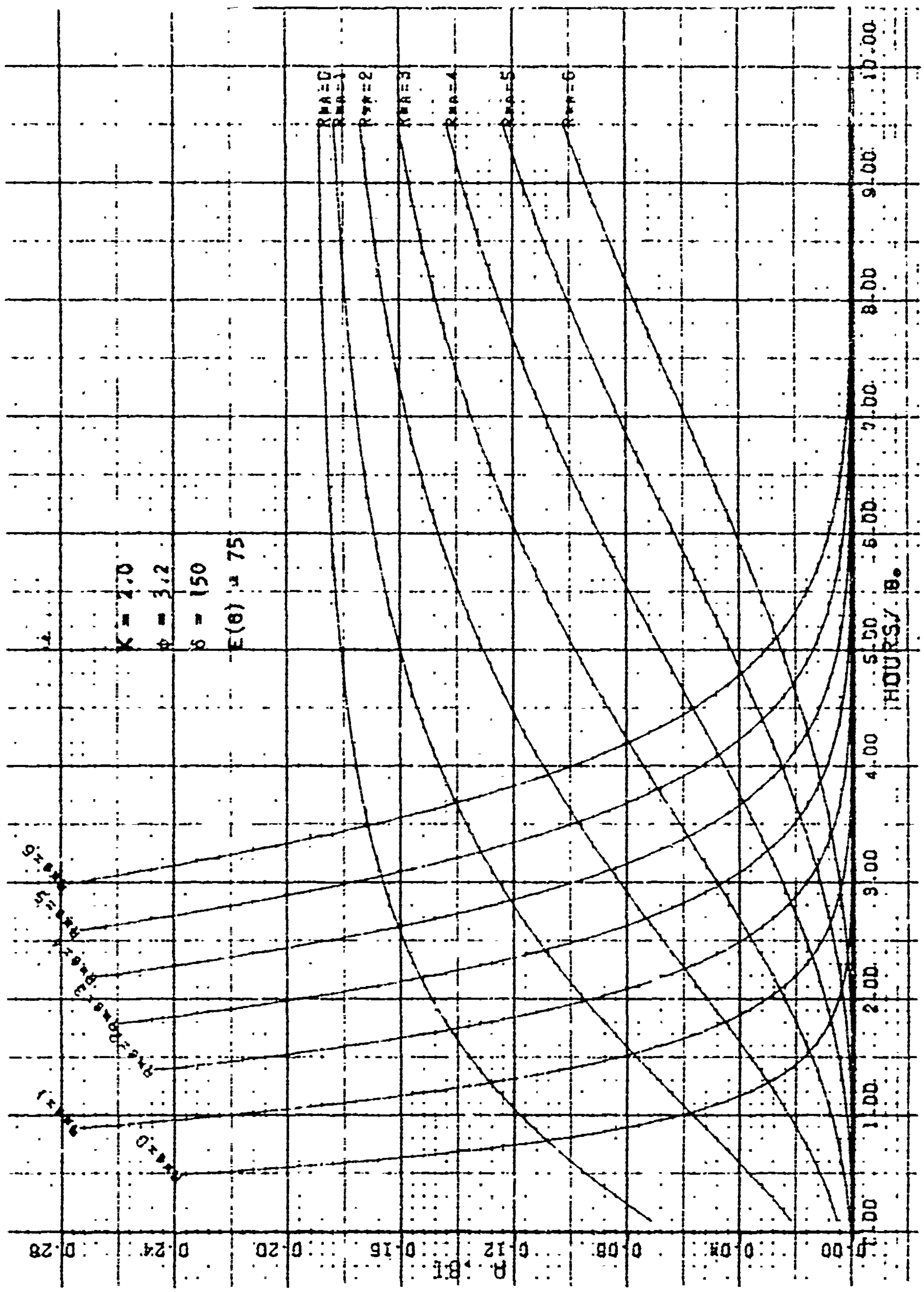

FIGURE 3 - Bayes Method; Criteria Set A-B

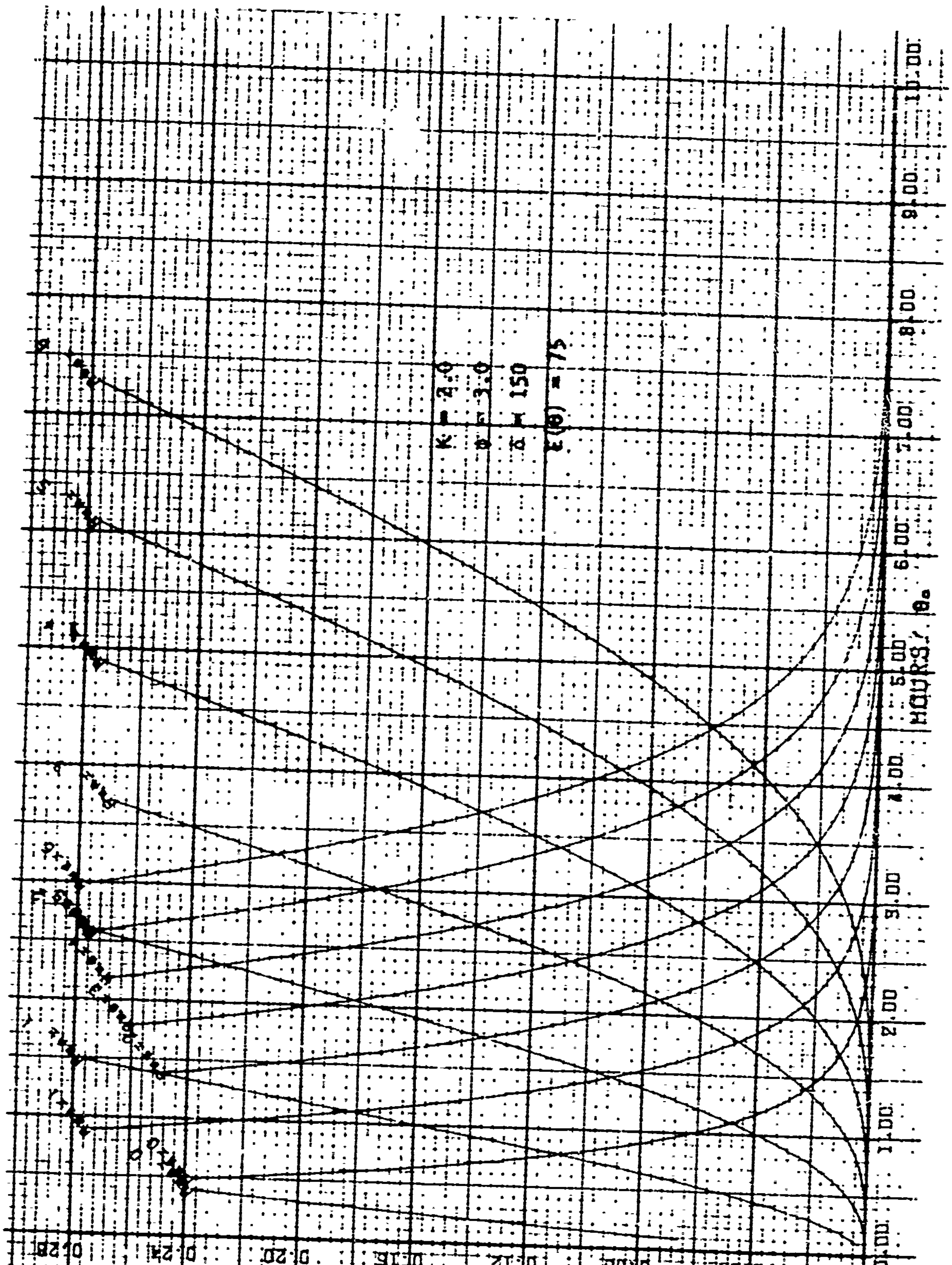

FIGURE 4 - Bayes Method; Criteria Set A_1-B_1

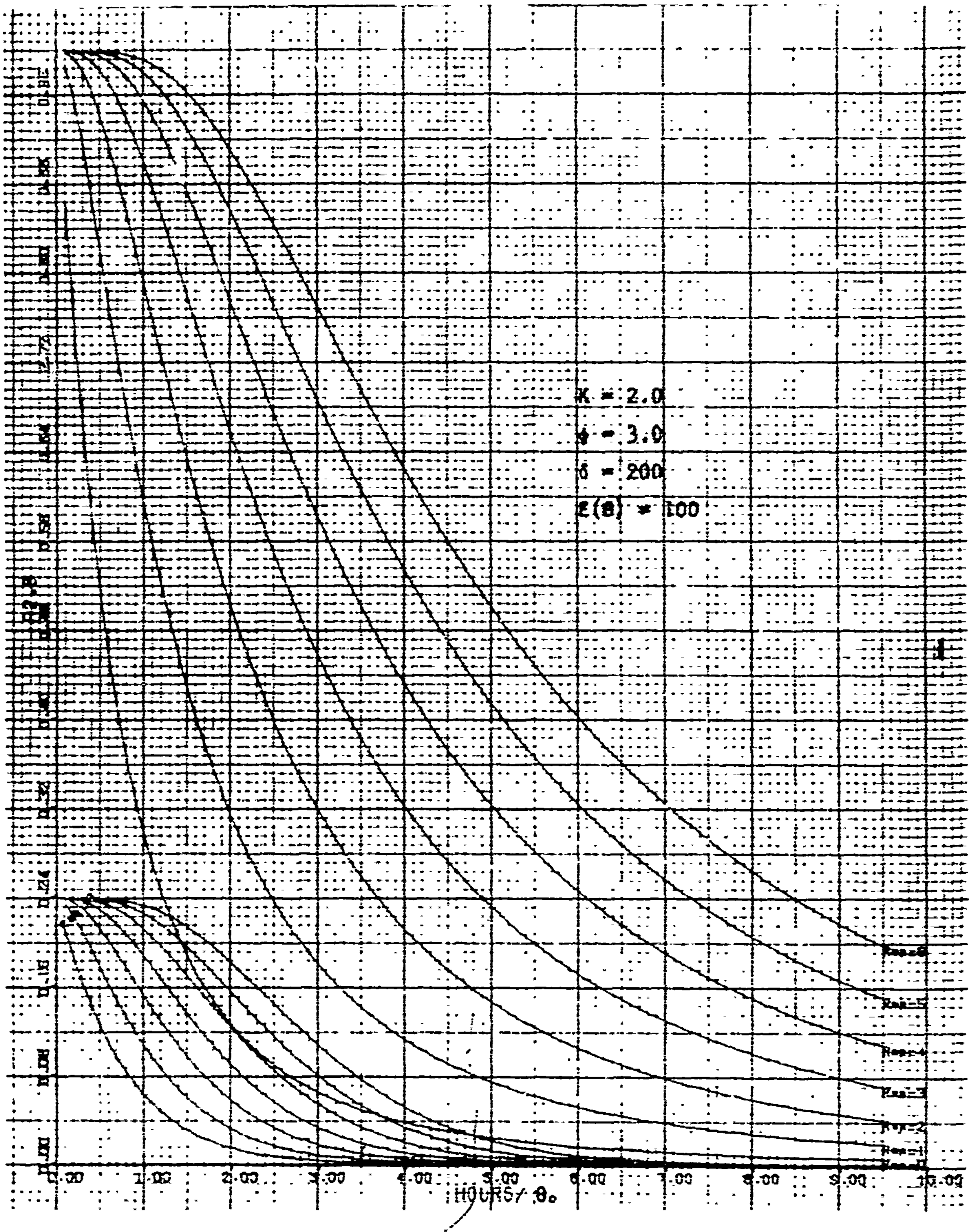

FIGURE 5 - Bayes Method; Criteria Set A_2-B

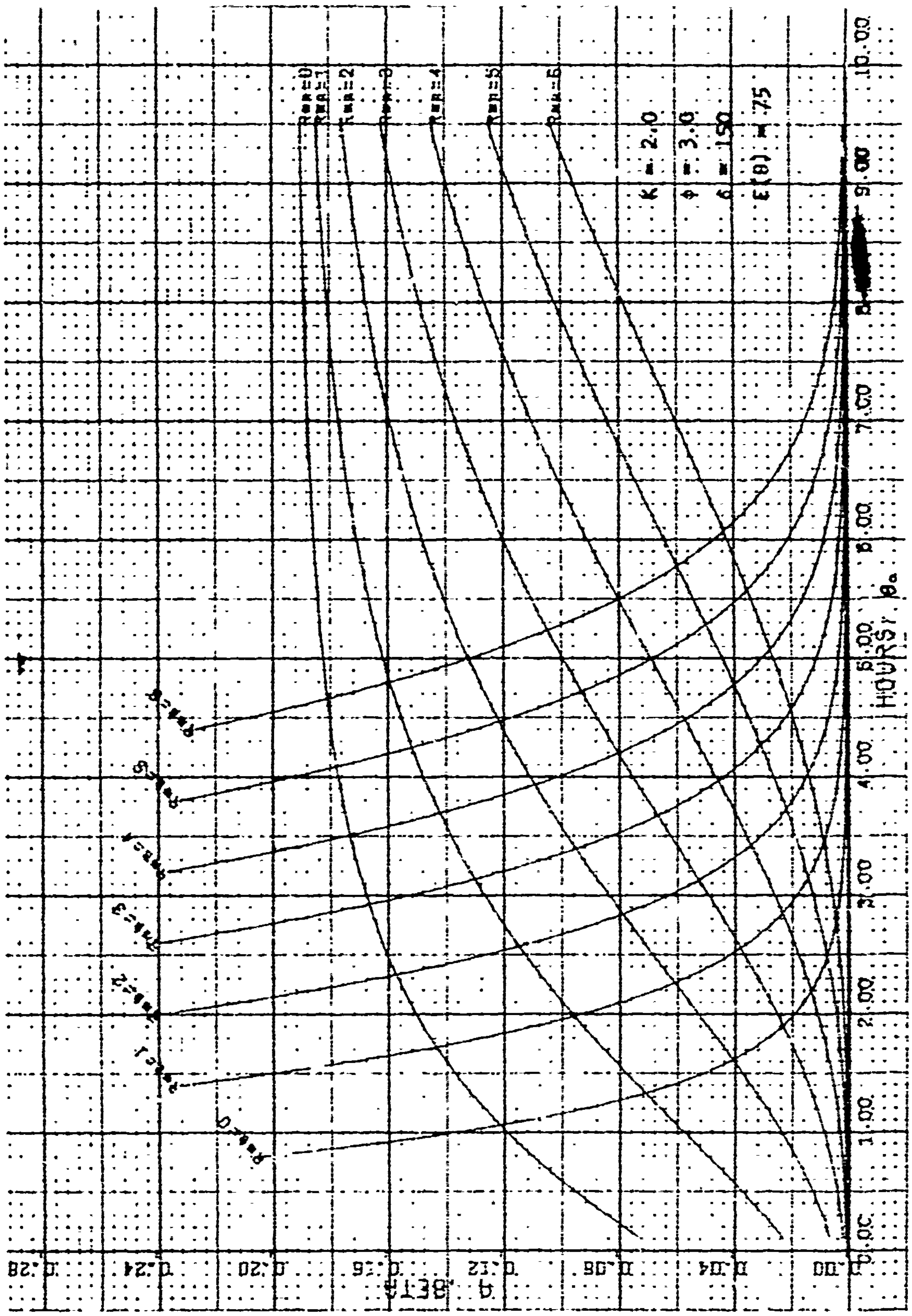

FIGURE 6 - Hybrid Method Criteria Set A-β

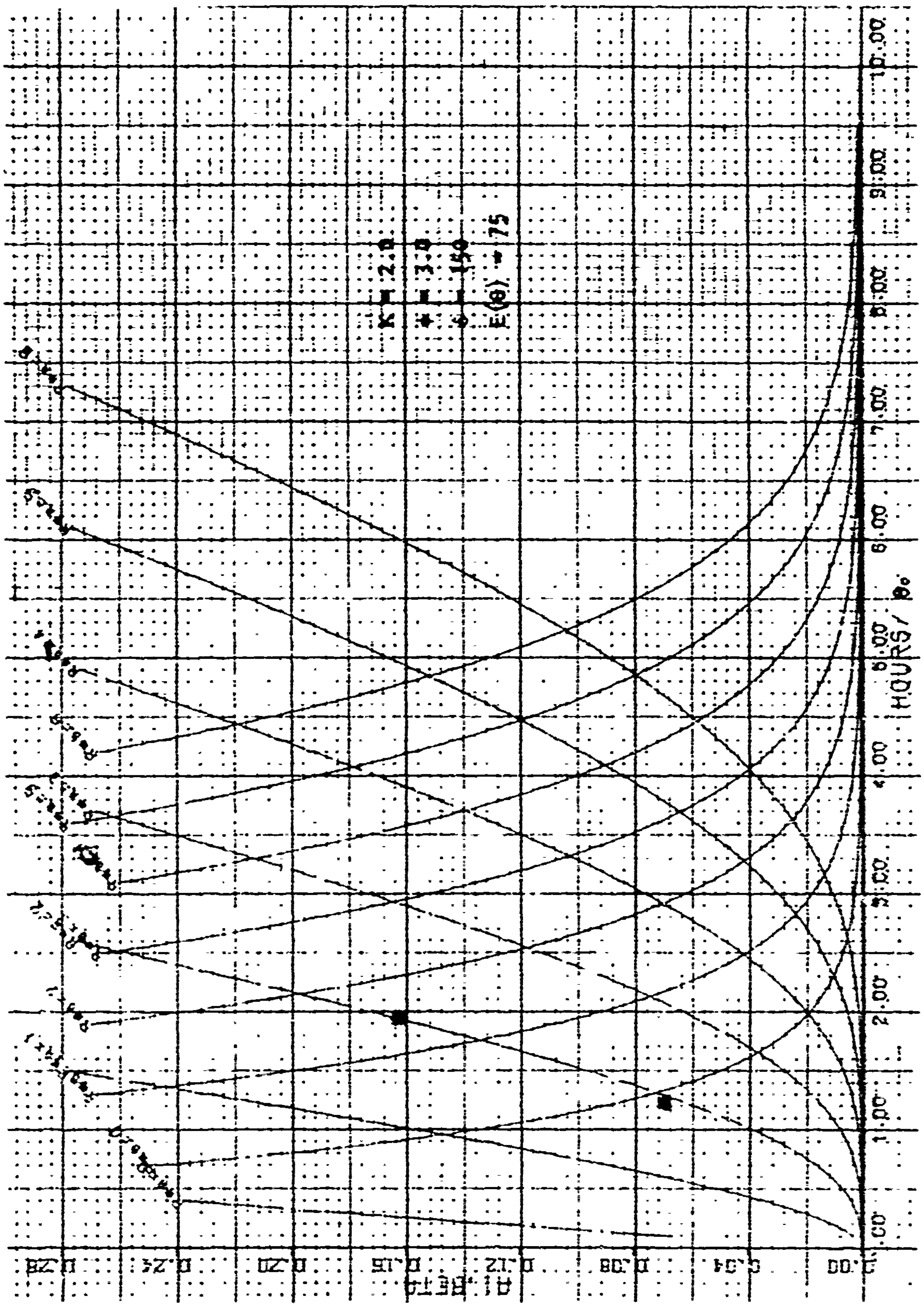

FIGURE 7 - Hybrid Method; Criteria Set A_1-β

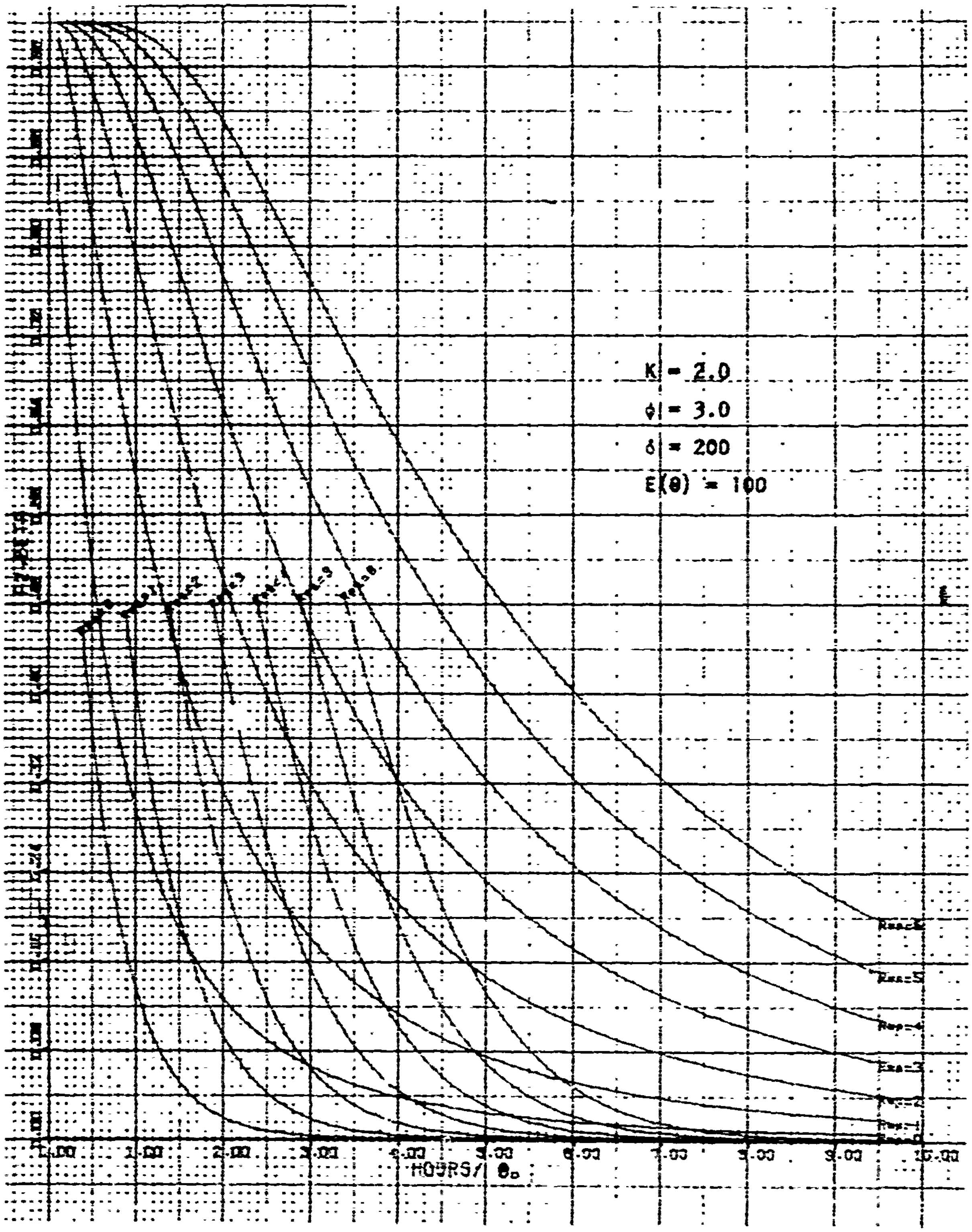

FIGURE 8 - Hybrid Method; Criteria Set A_2-β

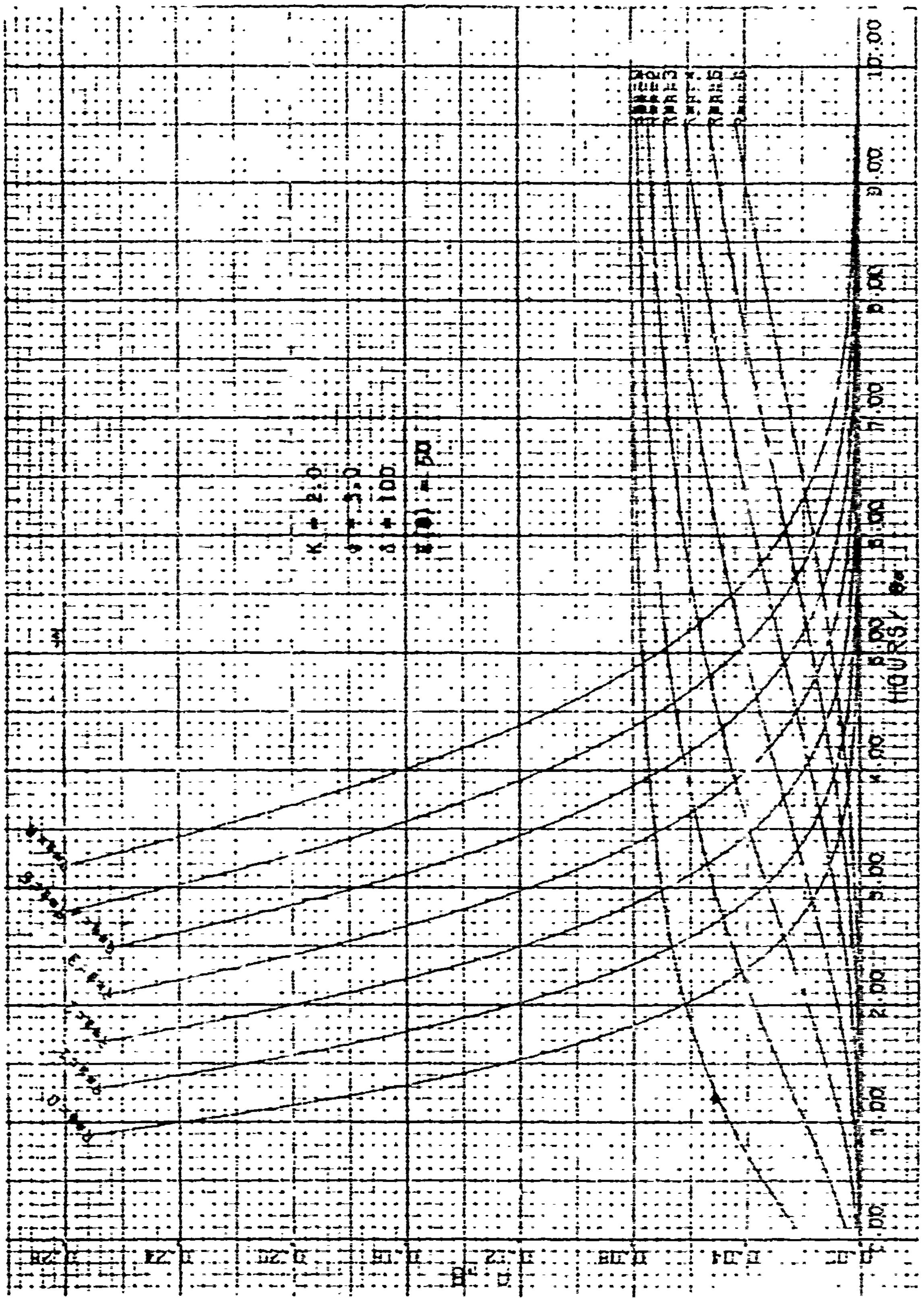

FIGURE 9 - Bayes Method; Criteria Set A-B

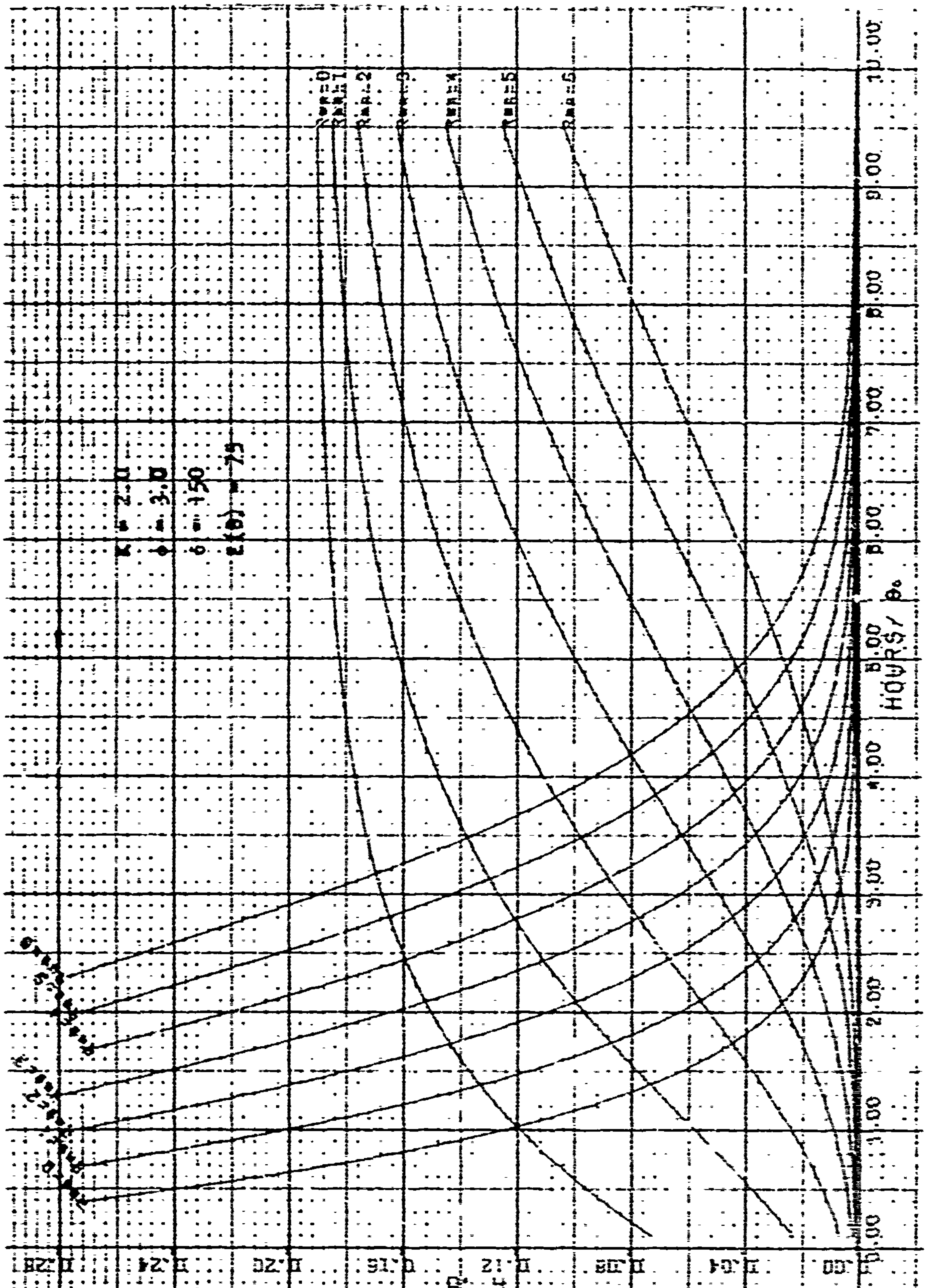

FIGURE 10 - Bayes Method; Criteria Set A-B

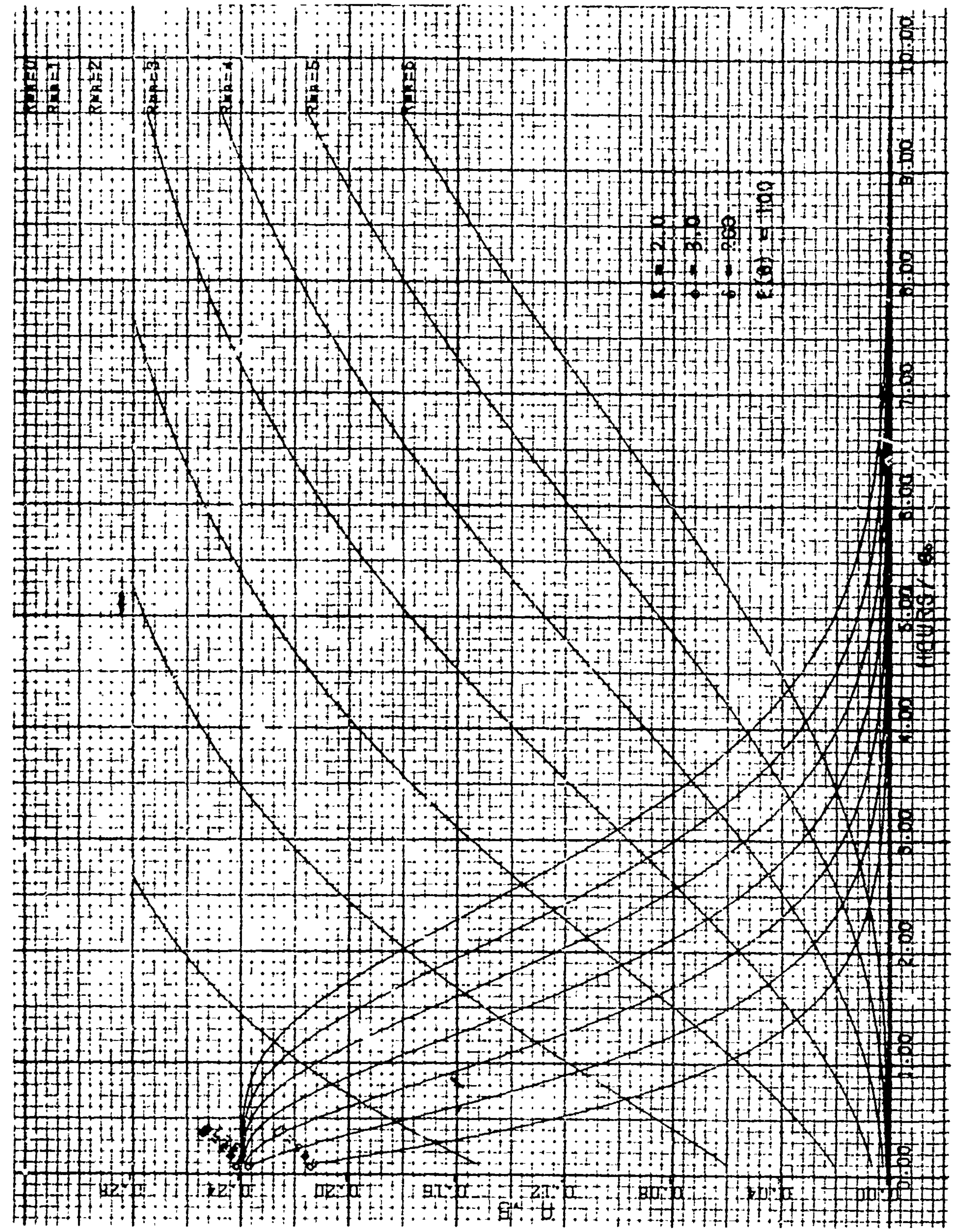

FIGURE 11 - Bayes Method; Criteria Set A-B

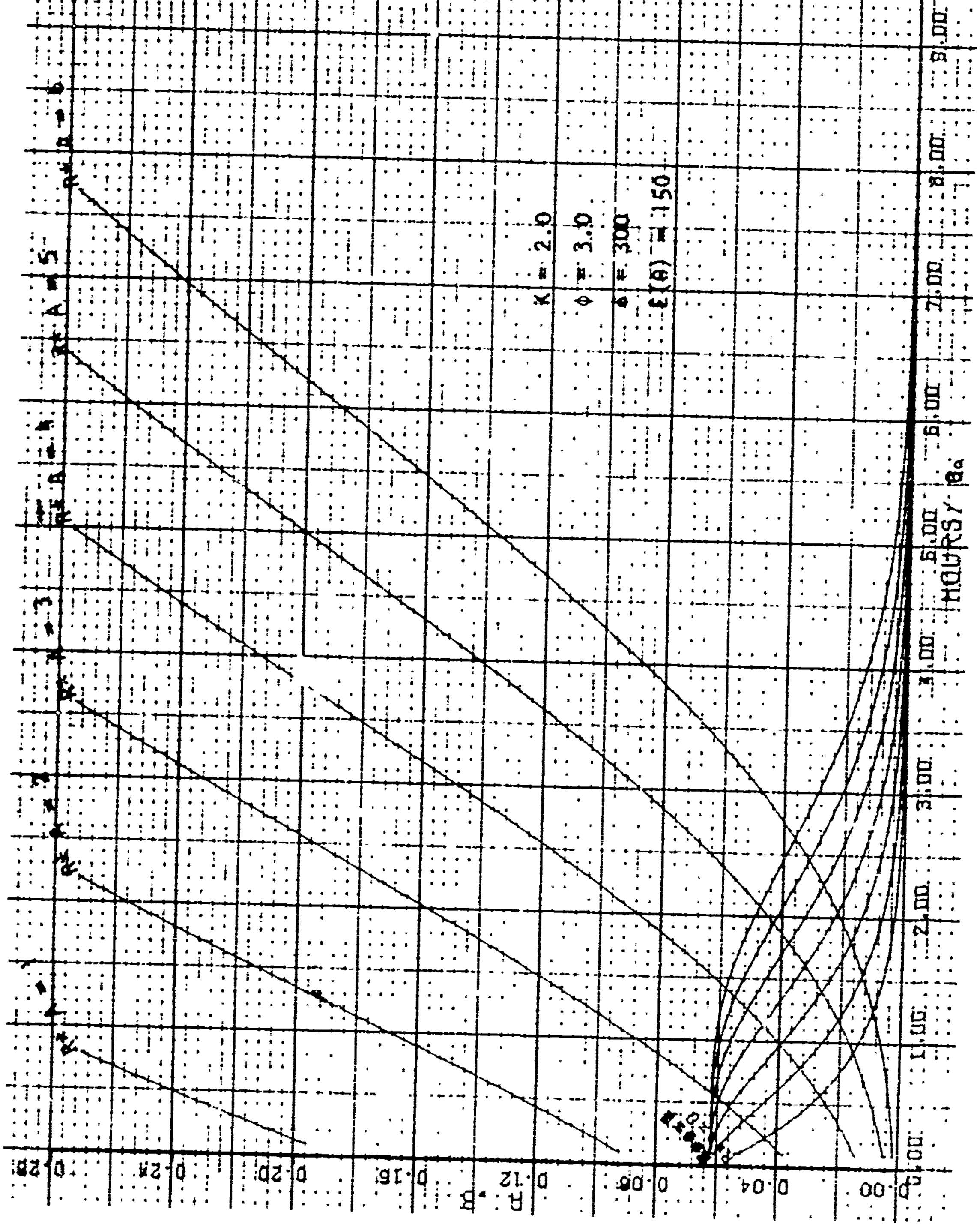

FIGURE 12 - Bayes Method; Criteria Set A-B

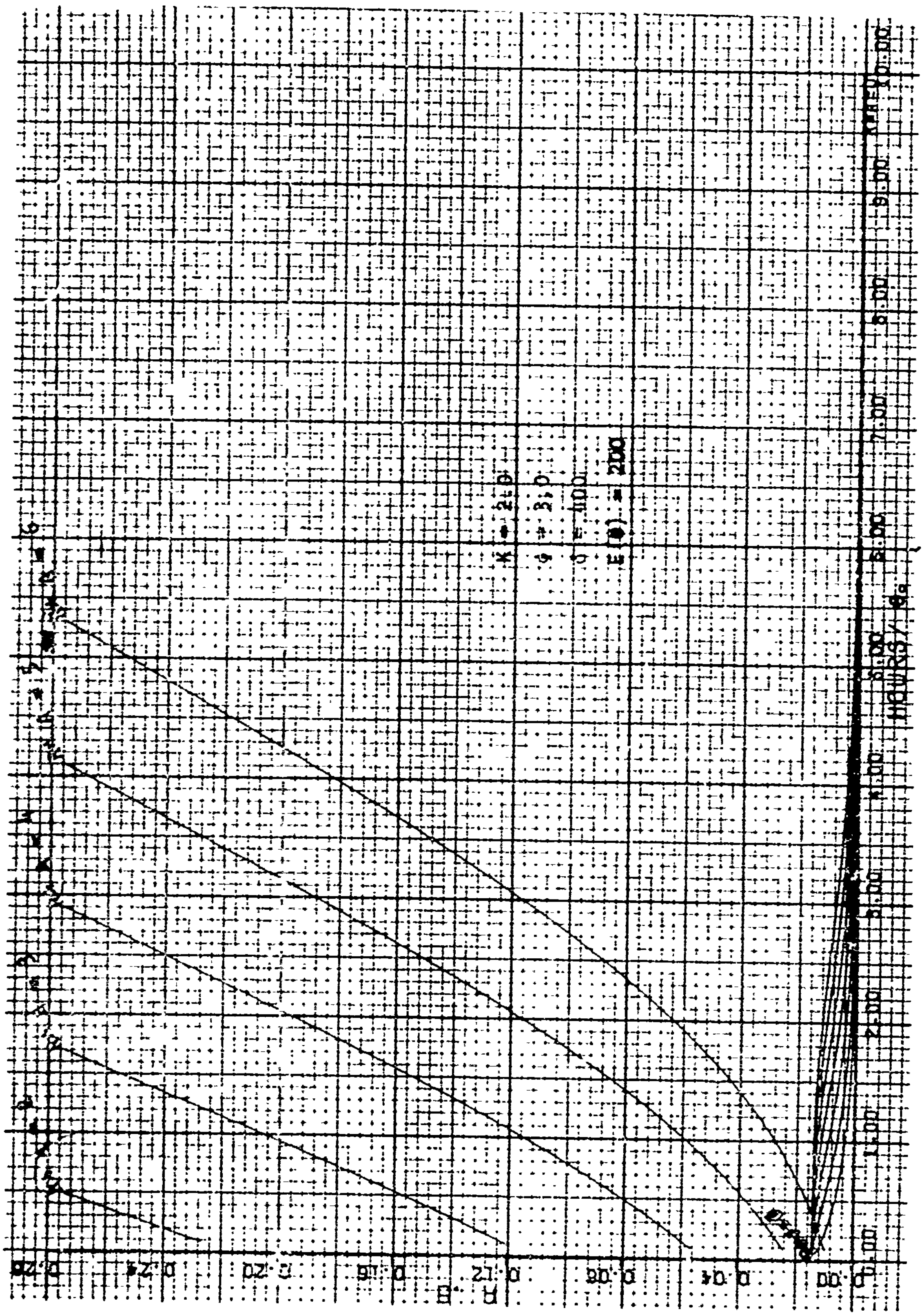

FIGURE 13 - Bayes Method; Criteria Set A-B

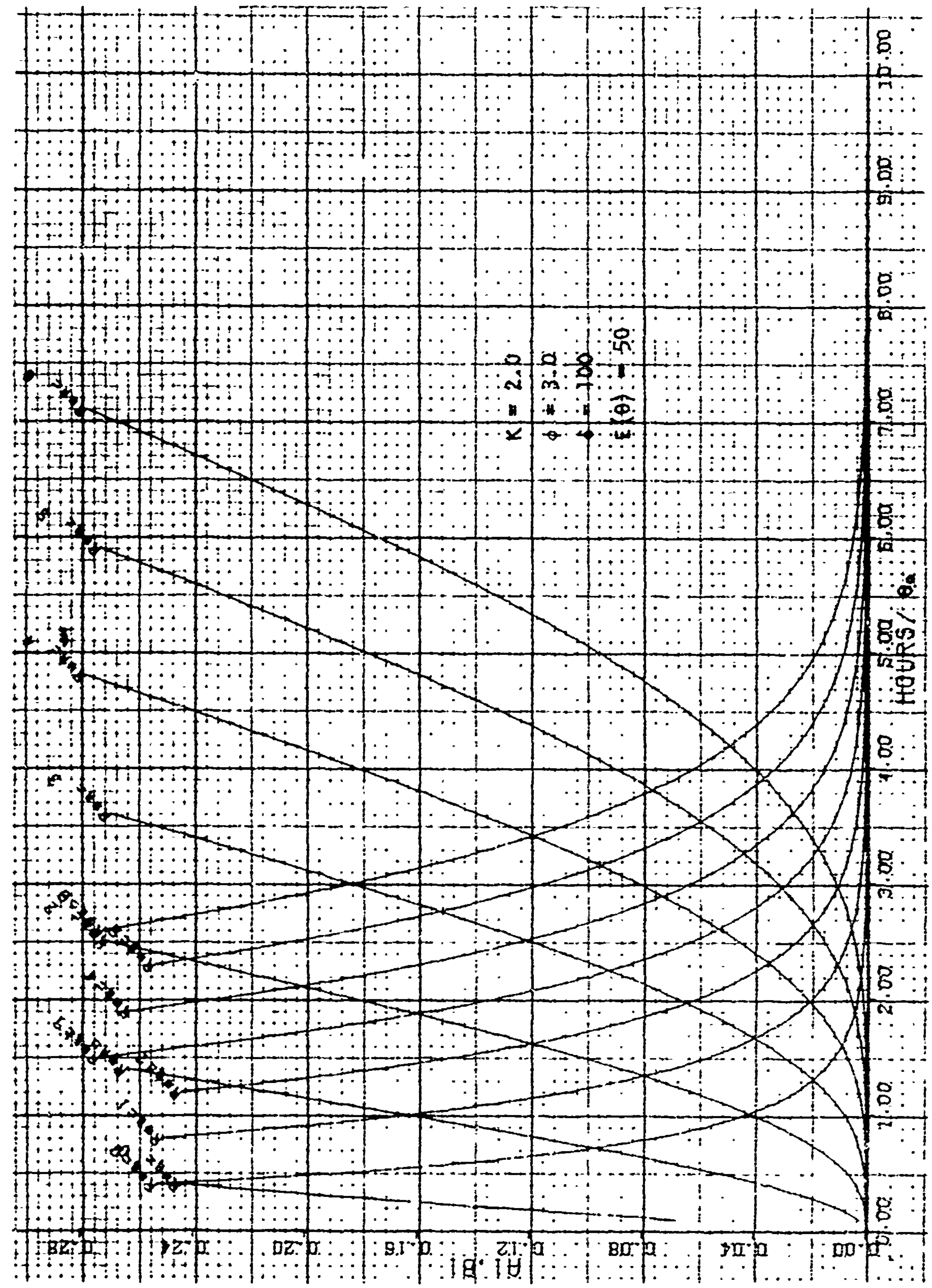

FIGURE 14 - Bayes Method; Criteria Set A_1-B_1

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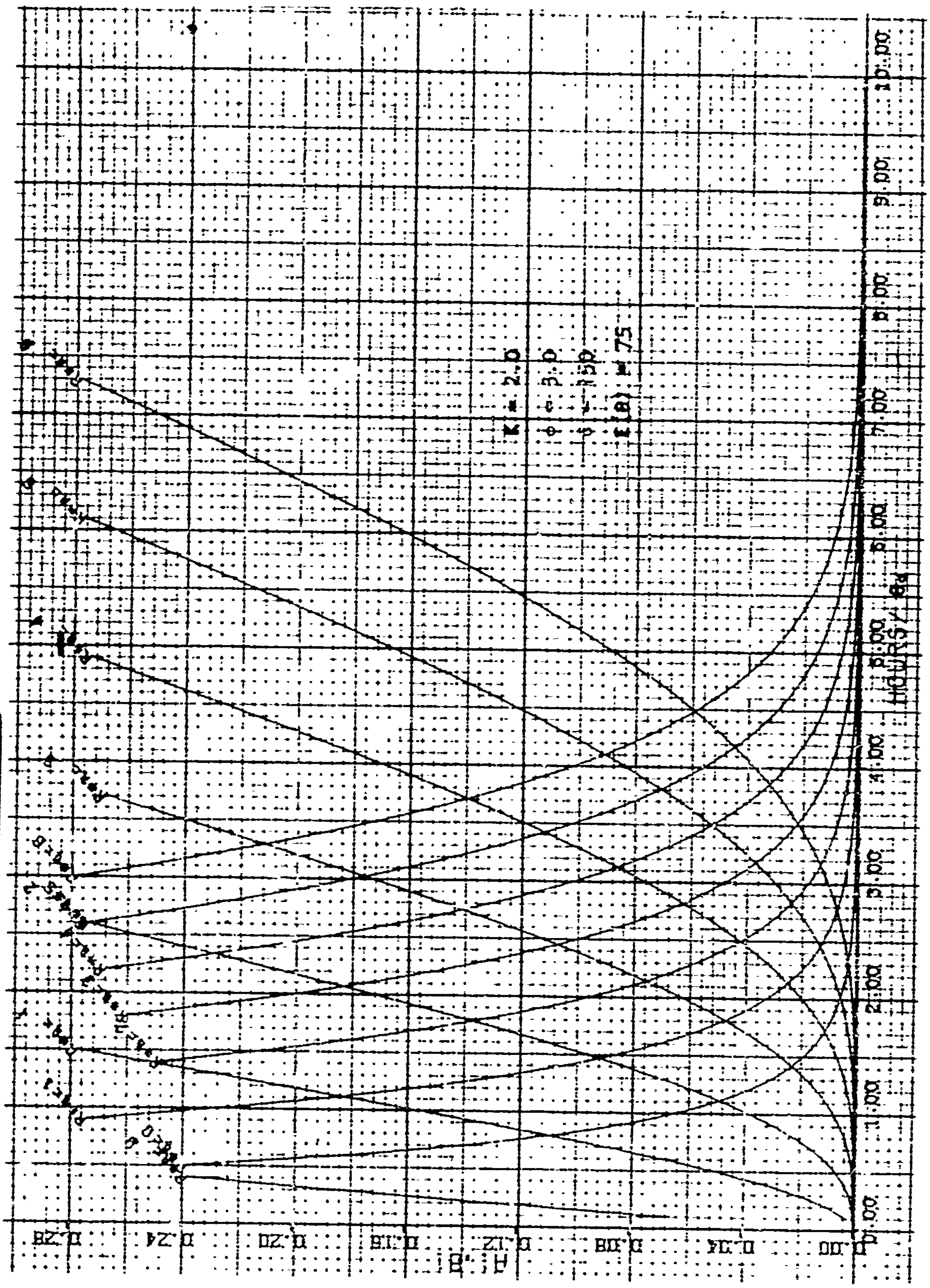

FIGURE 15 - Bayes Method; Criteria Set A_1-R_1

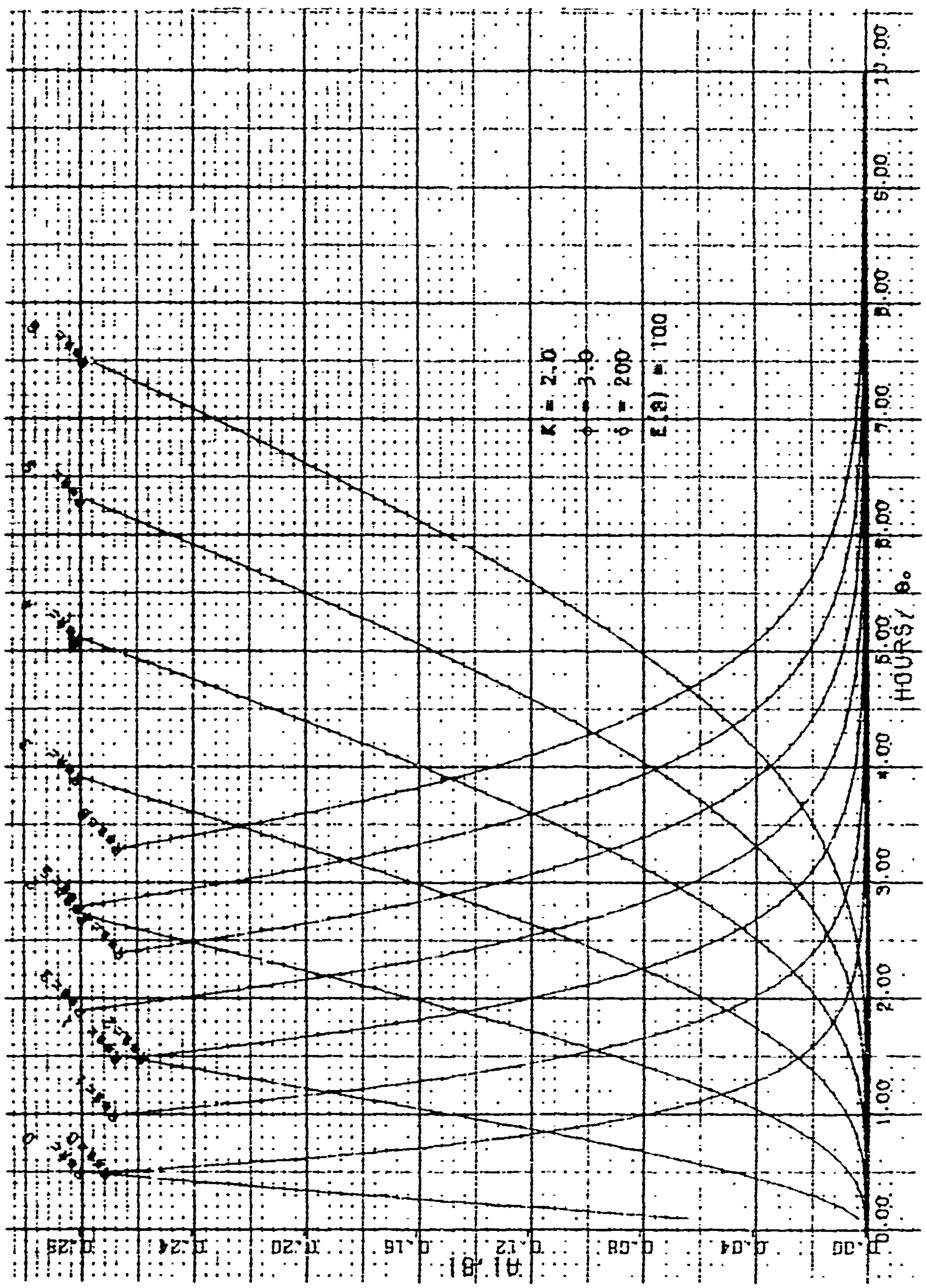

FIGURE 16 - Bayes Method; Criteria Set A_1-B_1

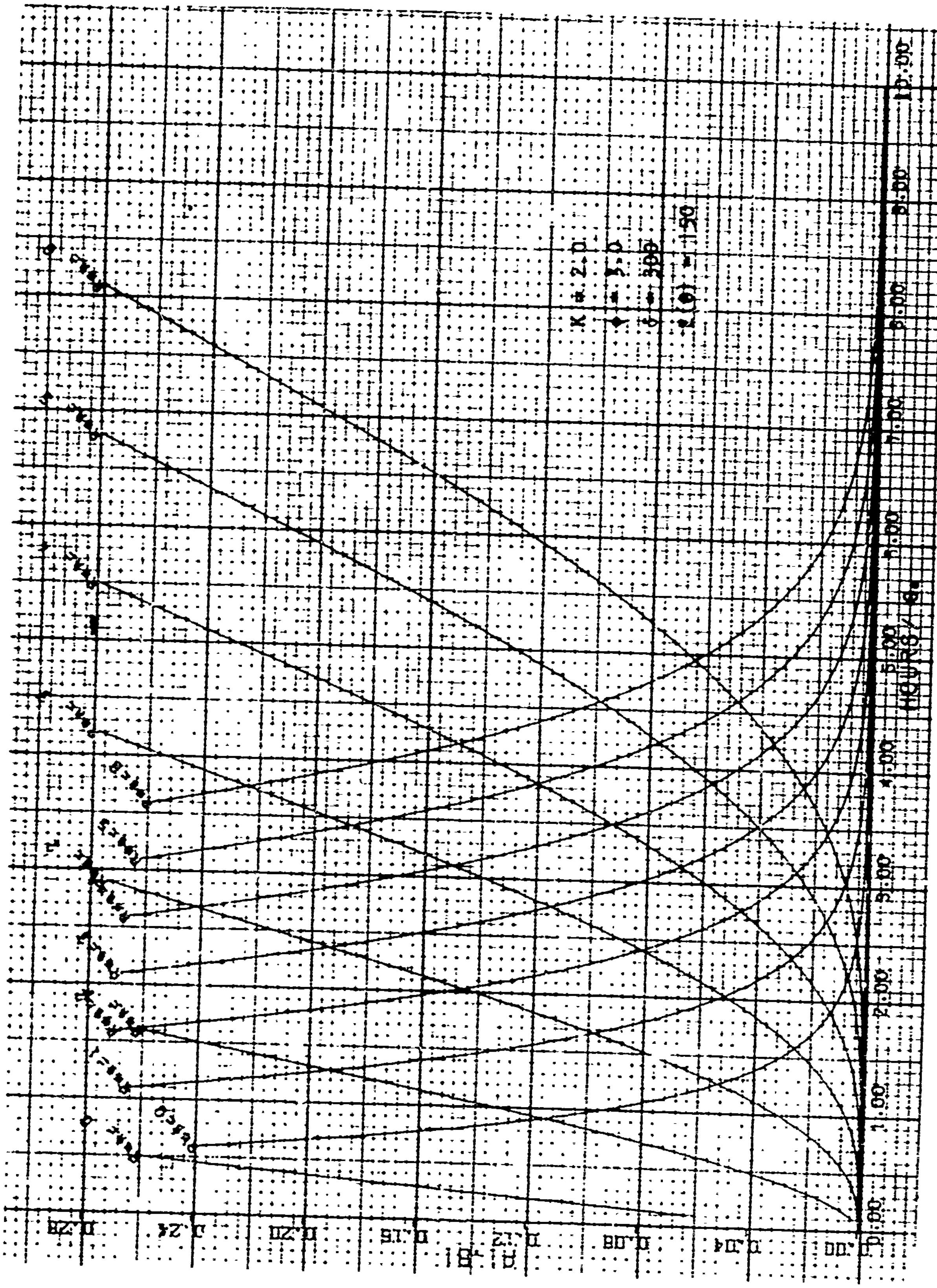

FIGURE 17 - Bayes Method; Criteria Set A_1-B_1

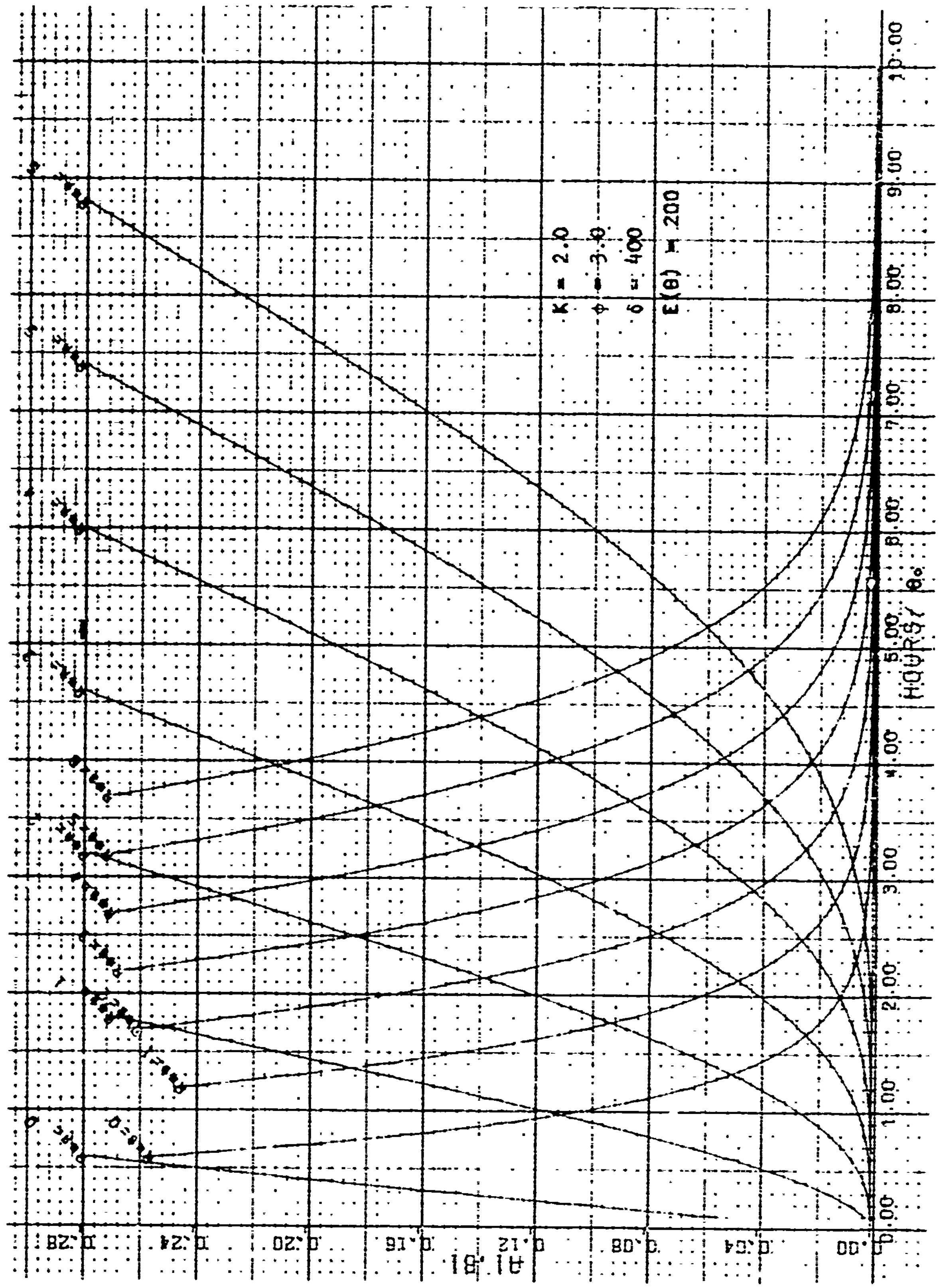

FIGURE 18 - Bayes Method; Criteria Set A_1-B_1

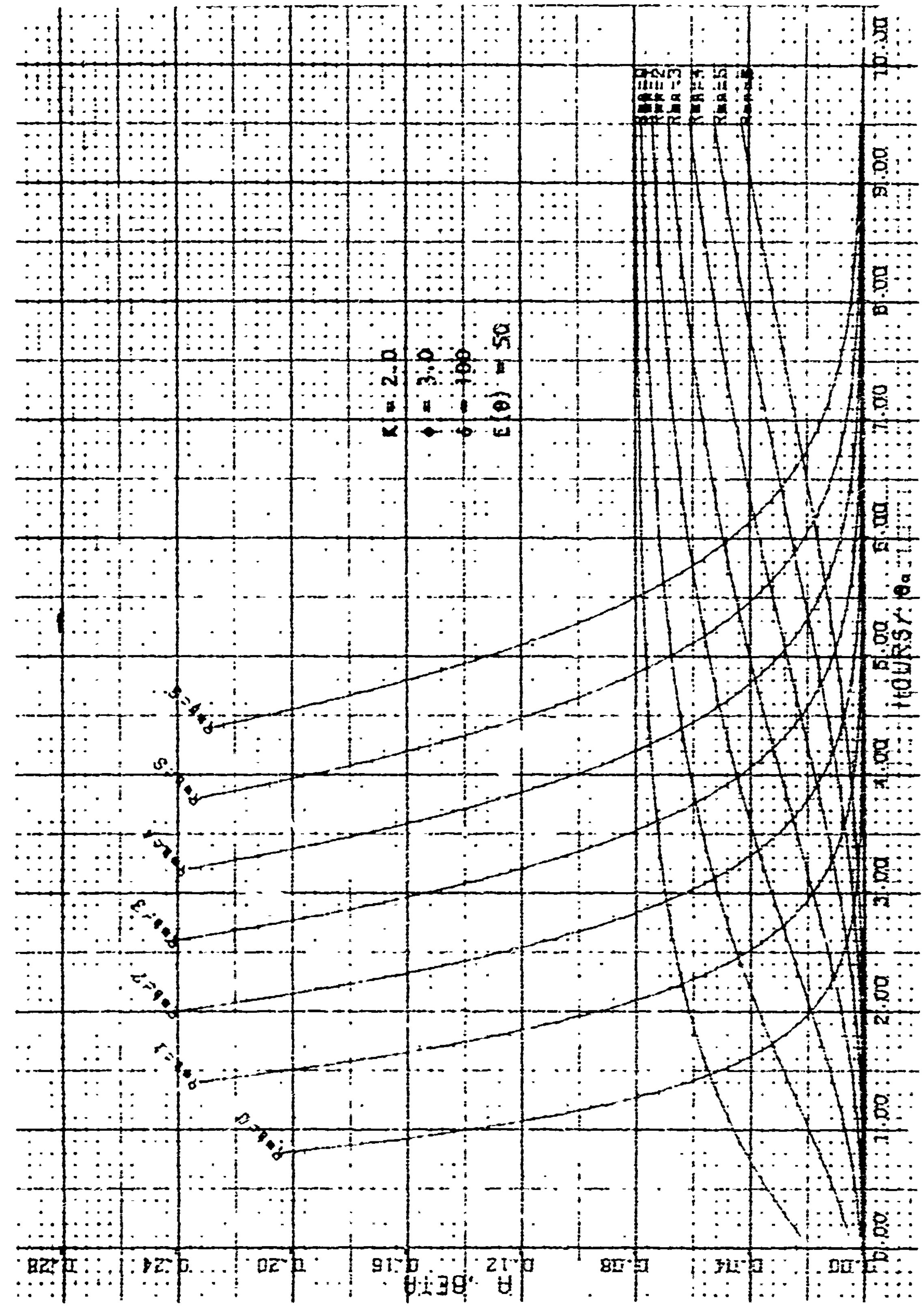

FIGURE 19 - Hybrid Method; Criteria Set A-β

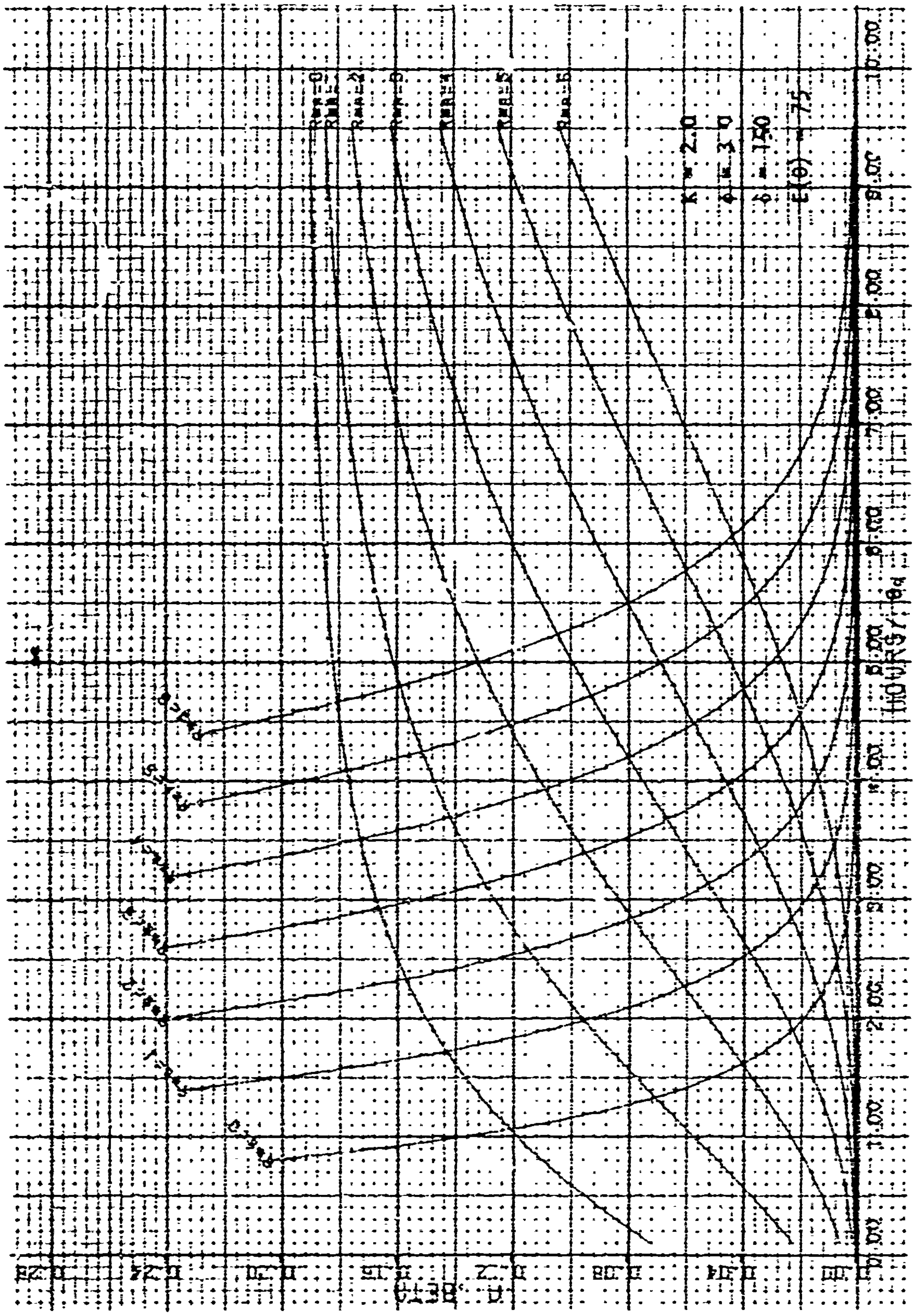

FIGURE 20 - Hybrid Method; Criteria Set A-β

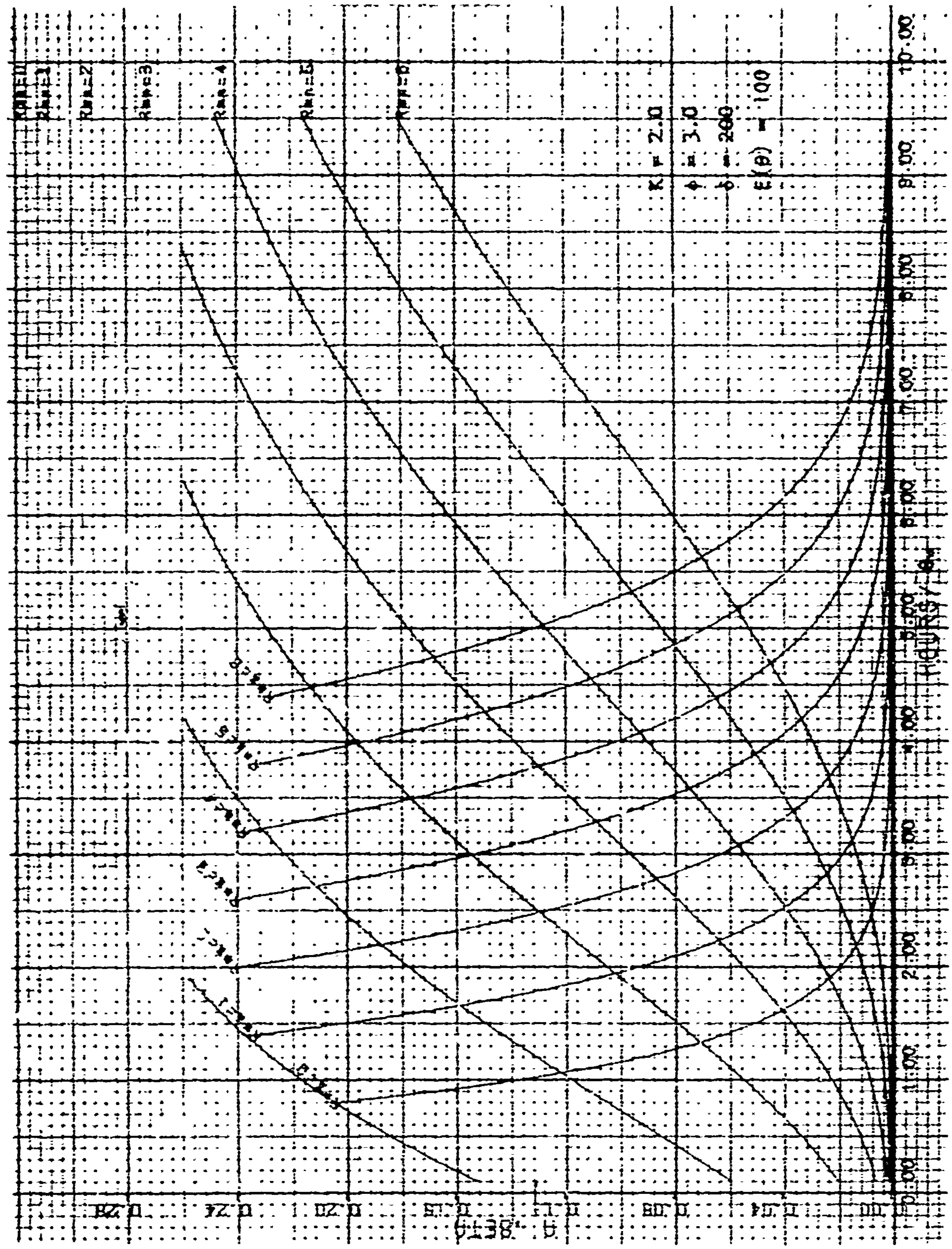

FIGURE 21 - Hybrid Method; Criteria Set A-β

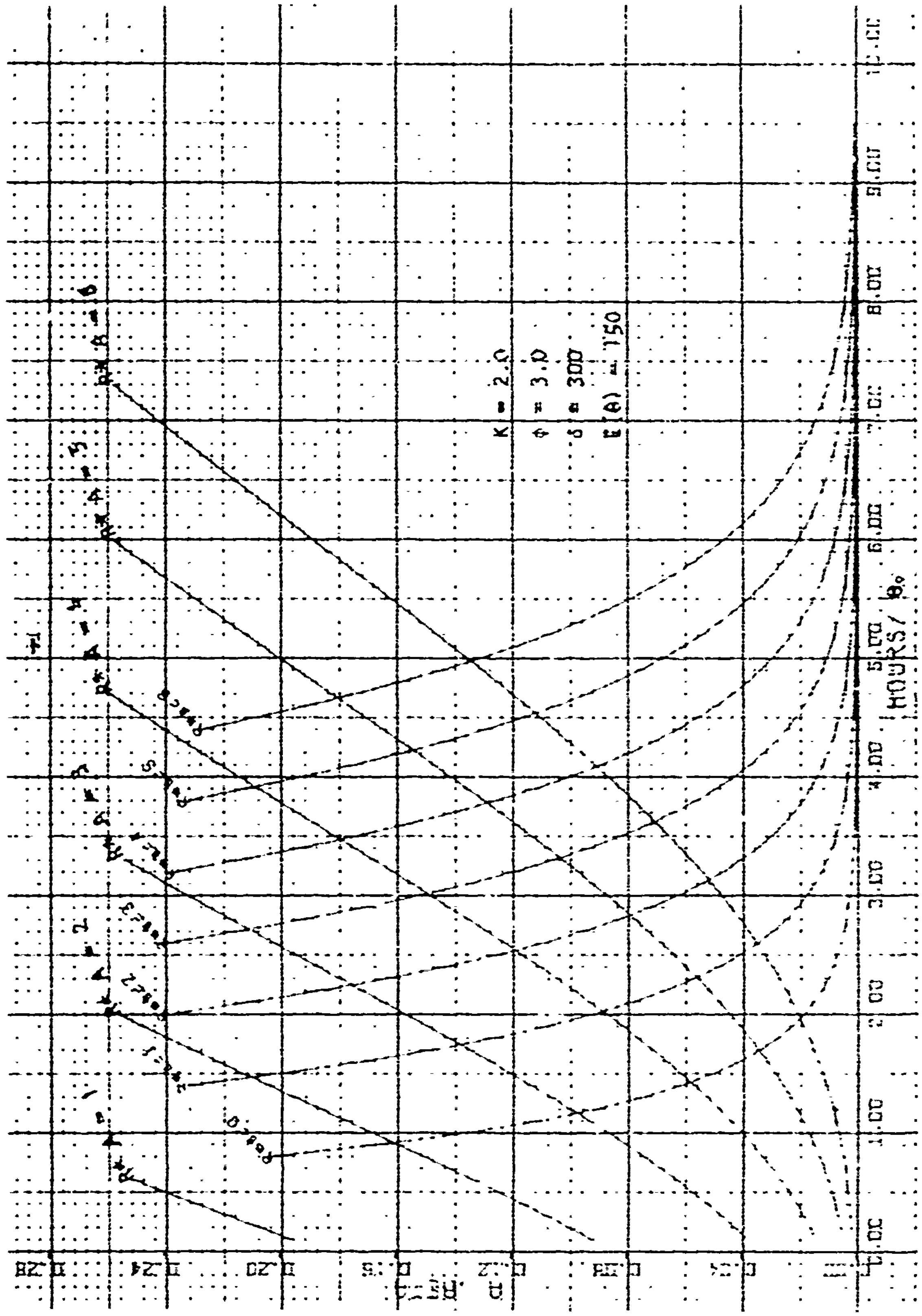

FIGURE 22 - Hybrid Method; Criteria Set A-β

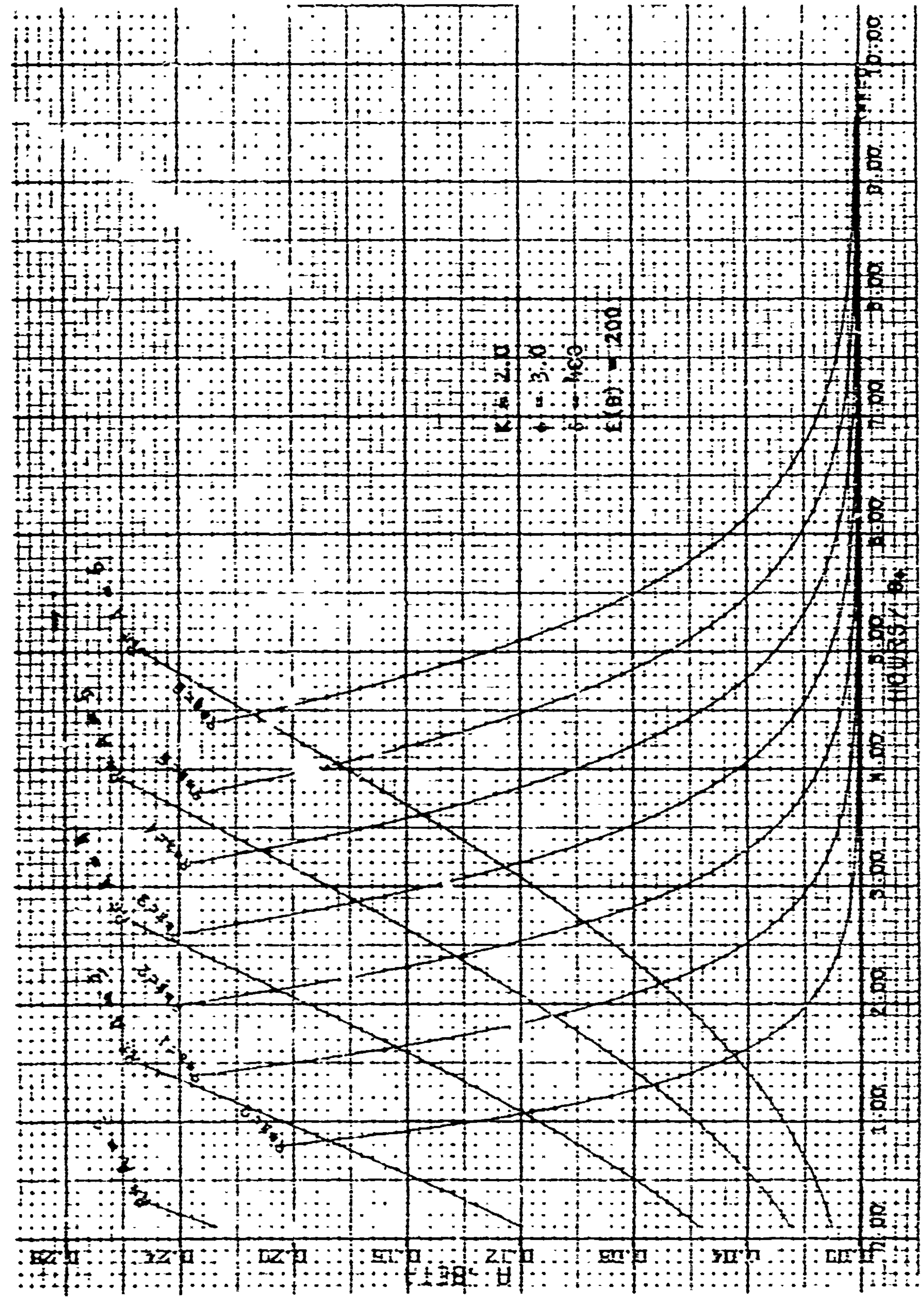

FIGURE 23 - Hybrid Method; Criteria Set A-β

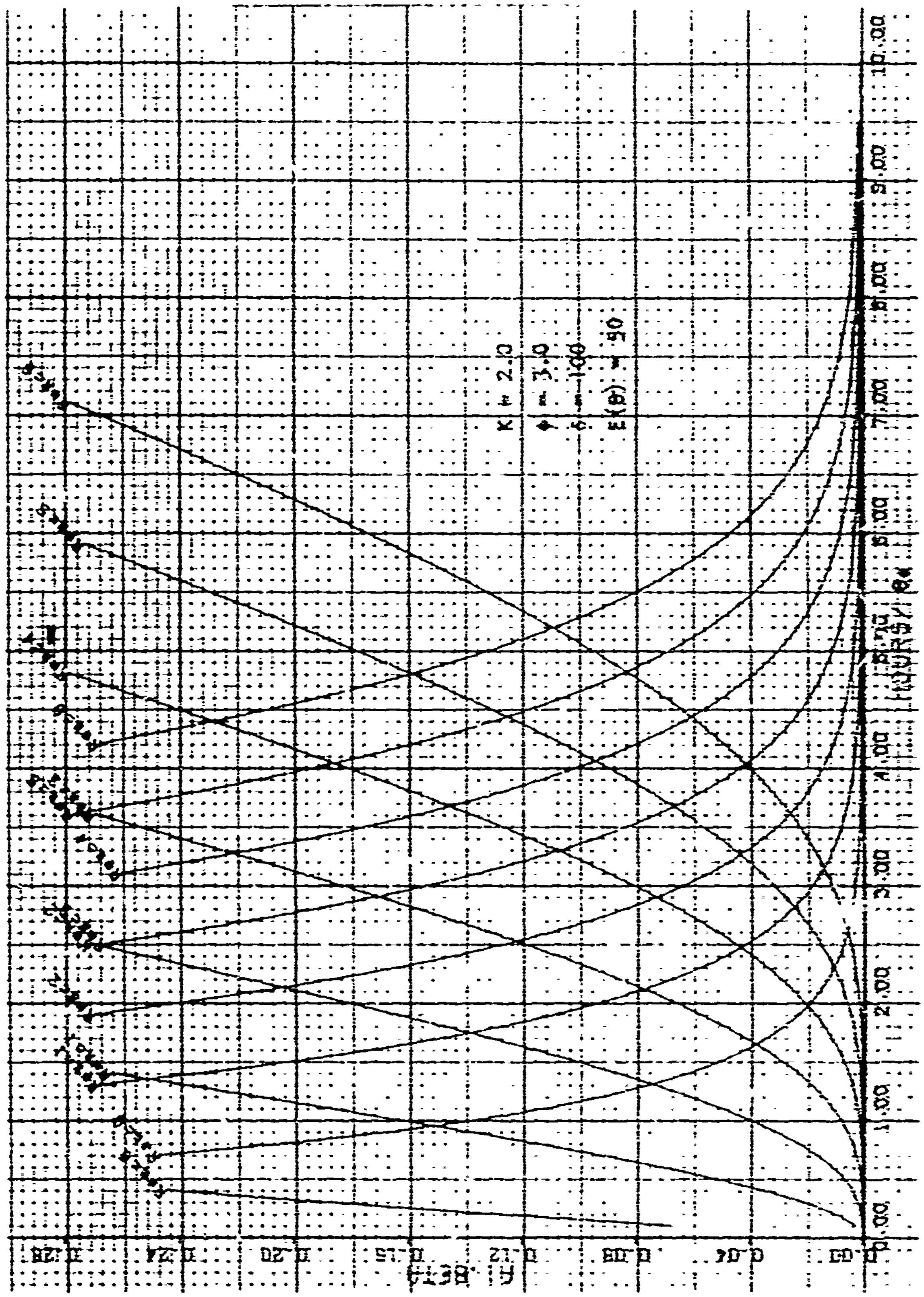

FIGURE 24 - Hybrid Method; Criteria Set Λ_1-β

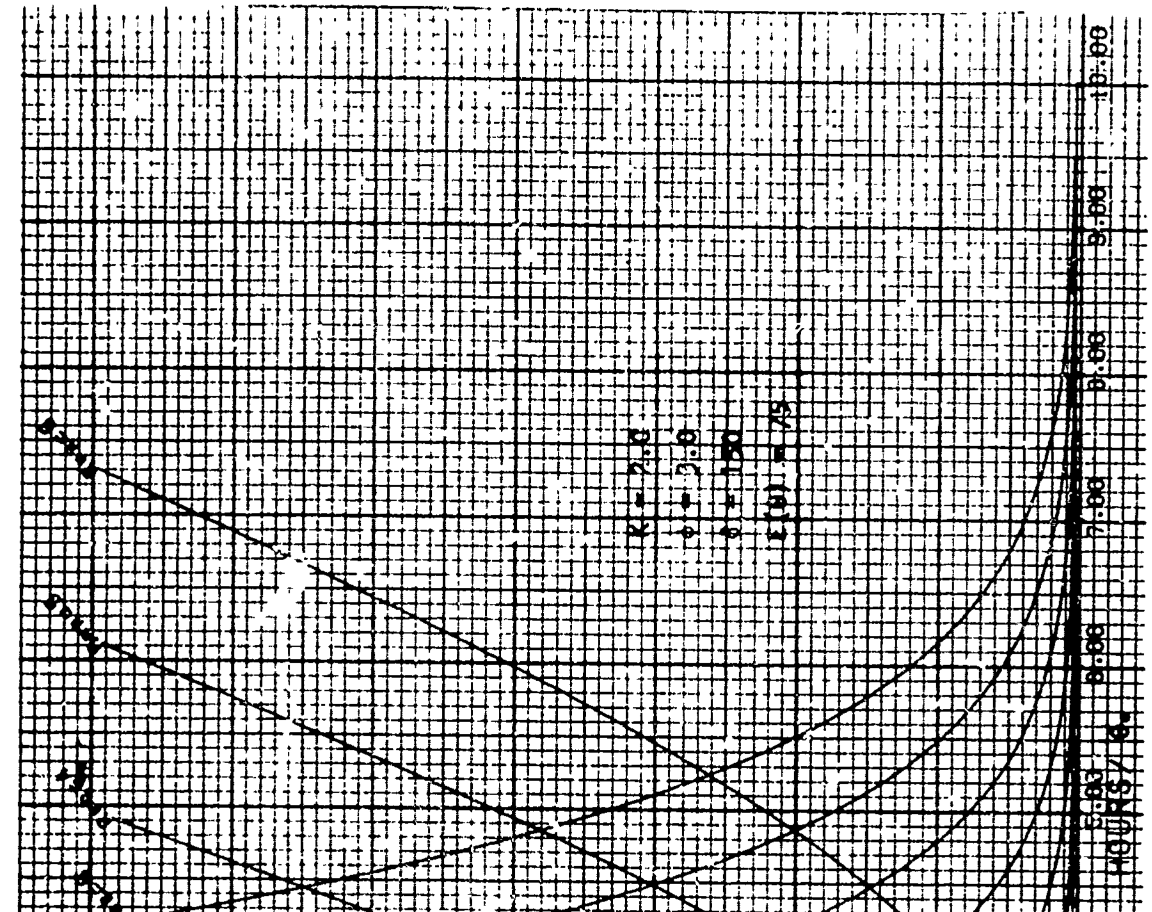

FIGURE 25 - Hybrid Method; Criteria Set A_1-β

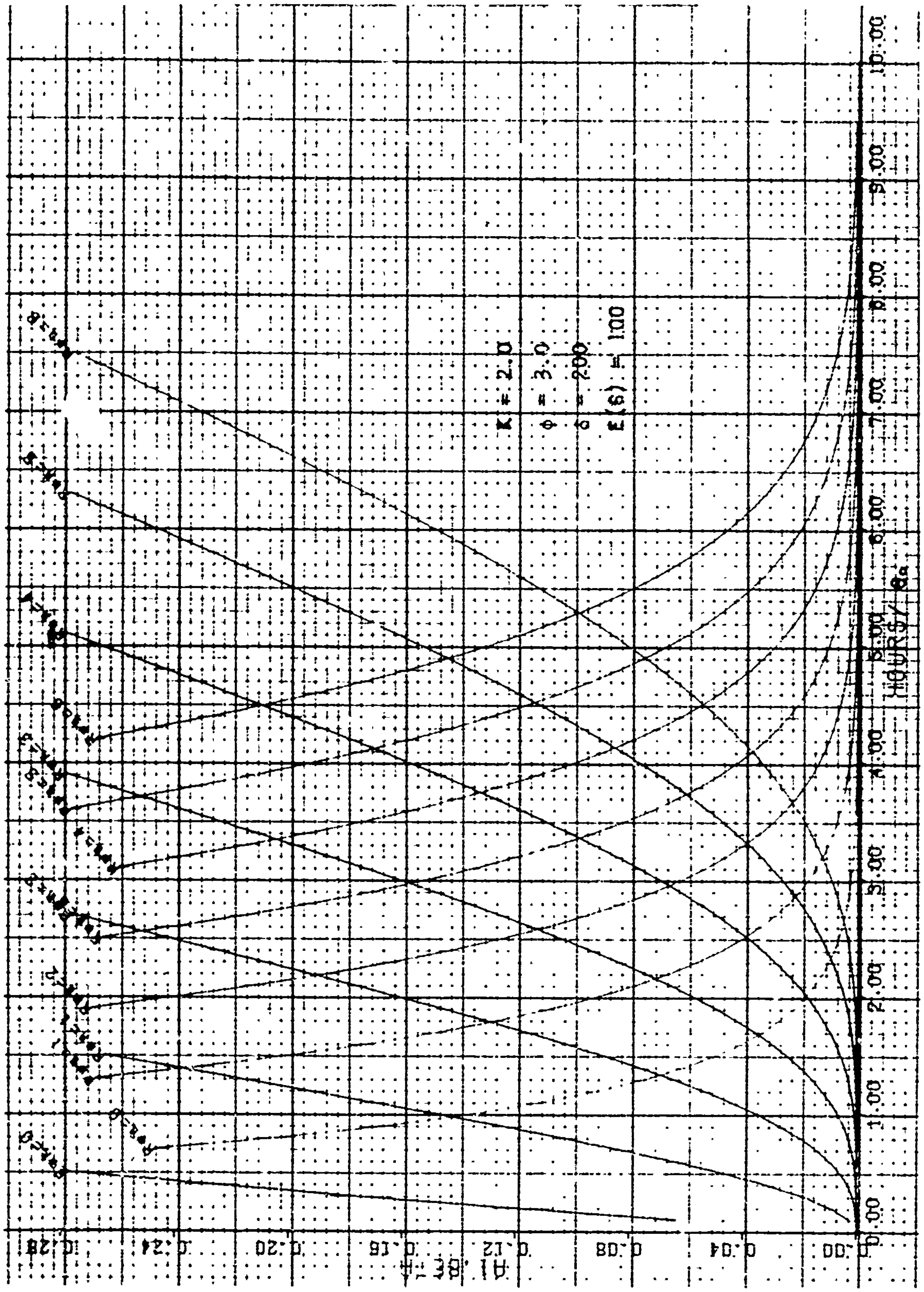

FIGURE 26 - Hybrid Method; Criteria Set A_1-B

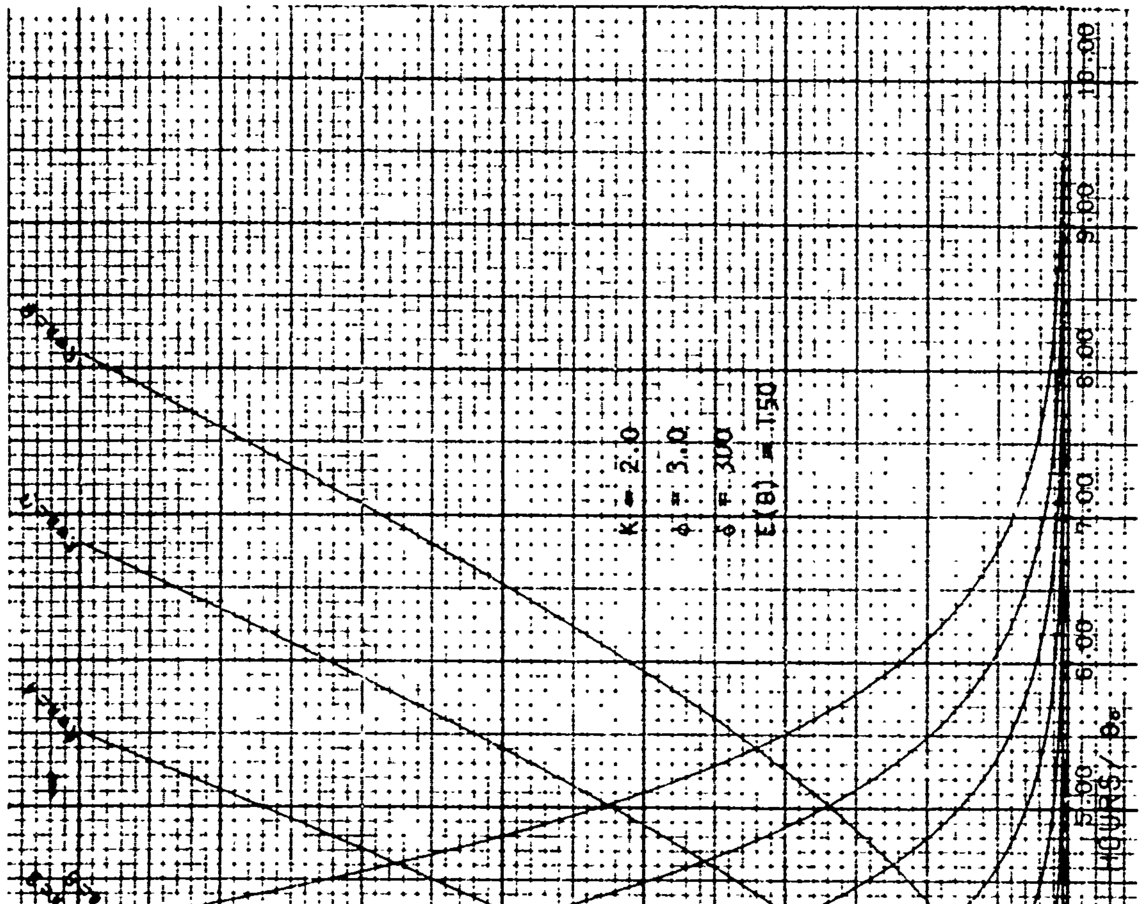

FIGURE 27 - Hybrid Method; Criteria Set A_1-β

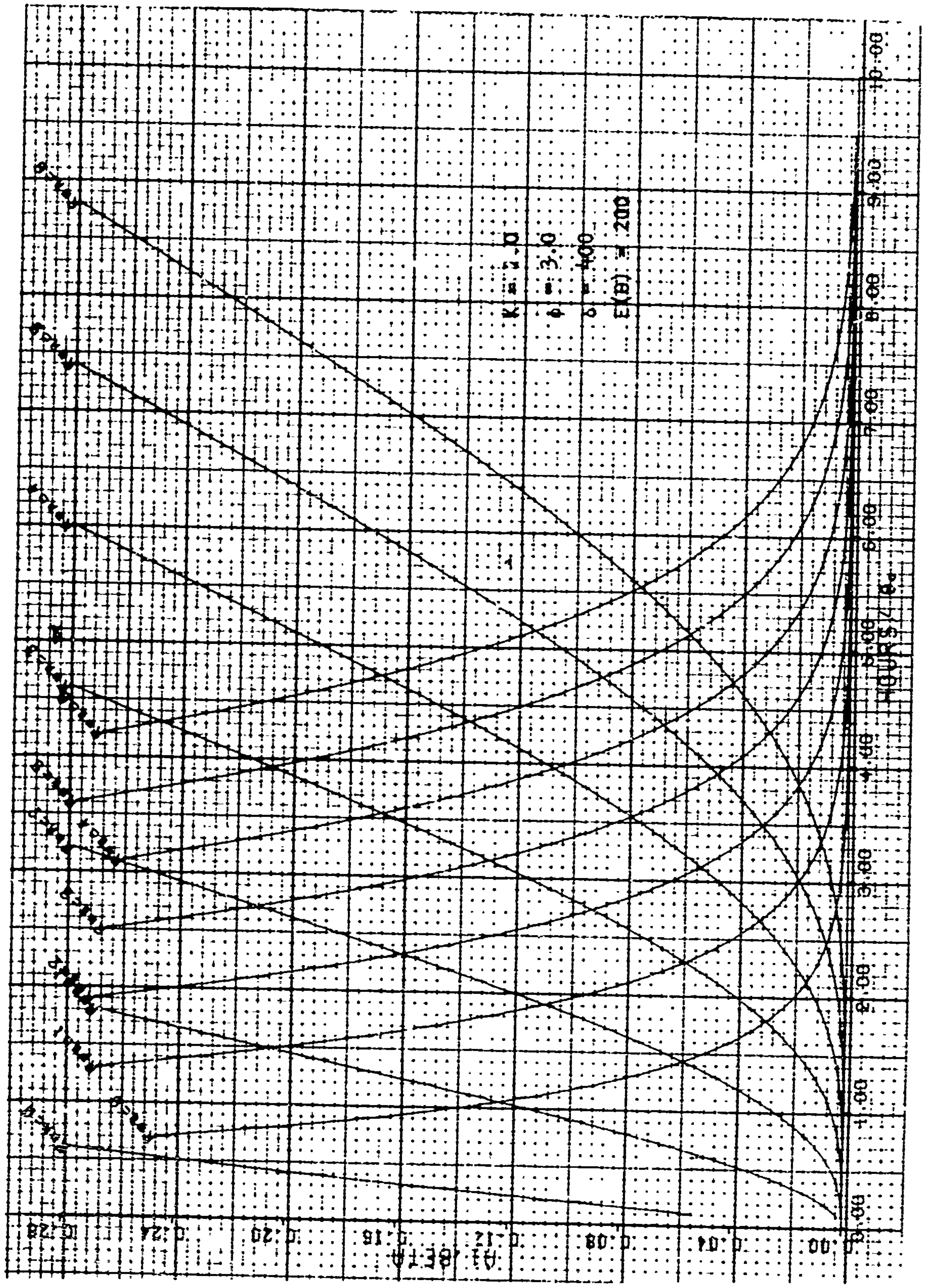

FIGURE 28 - Hybrid Method; Criteria Set Λ_1-β

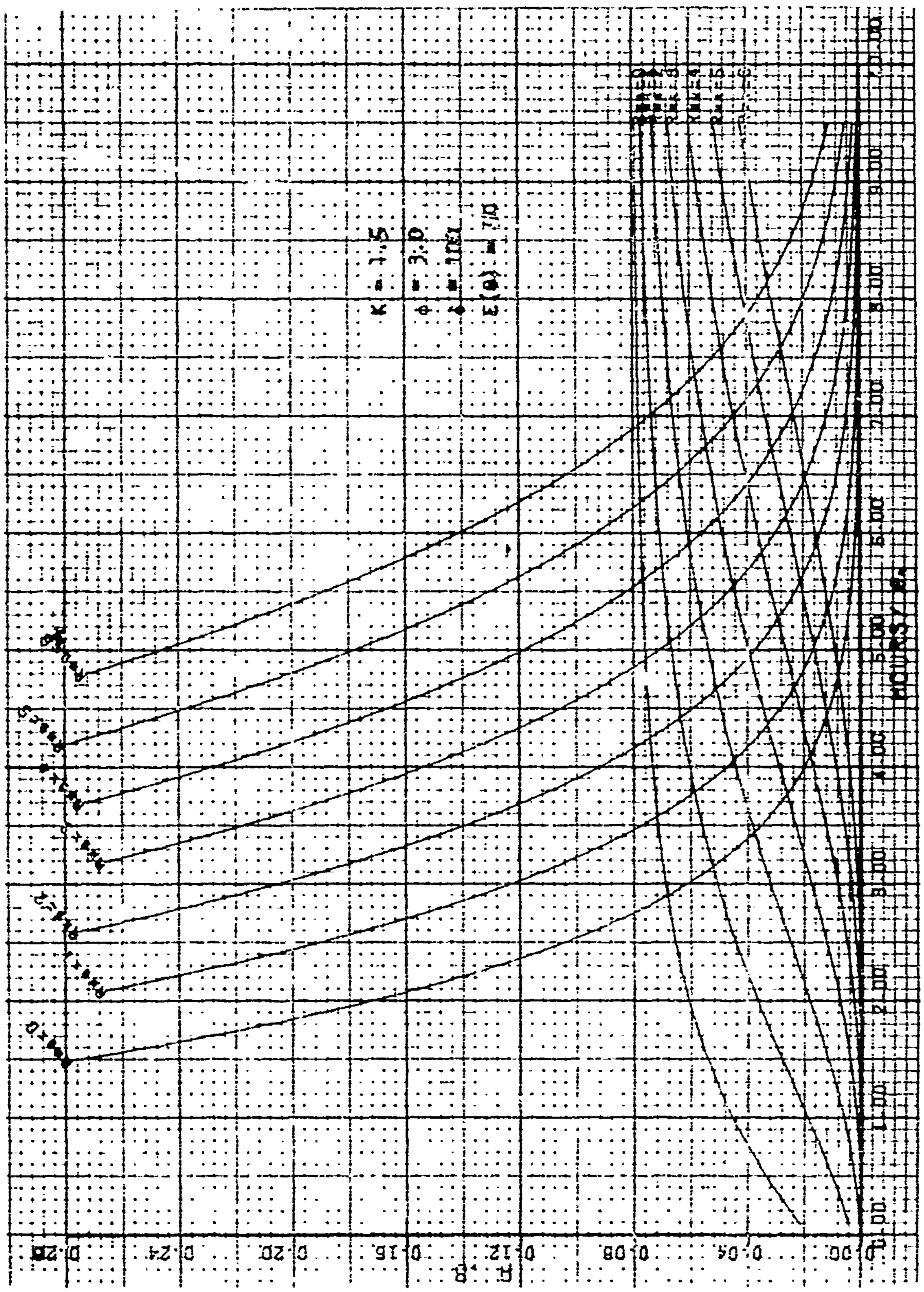

FIGURE 29 - Bayes Method; Criteria Set A-B

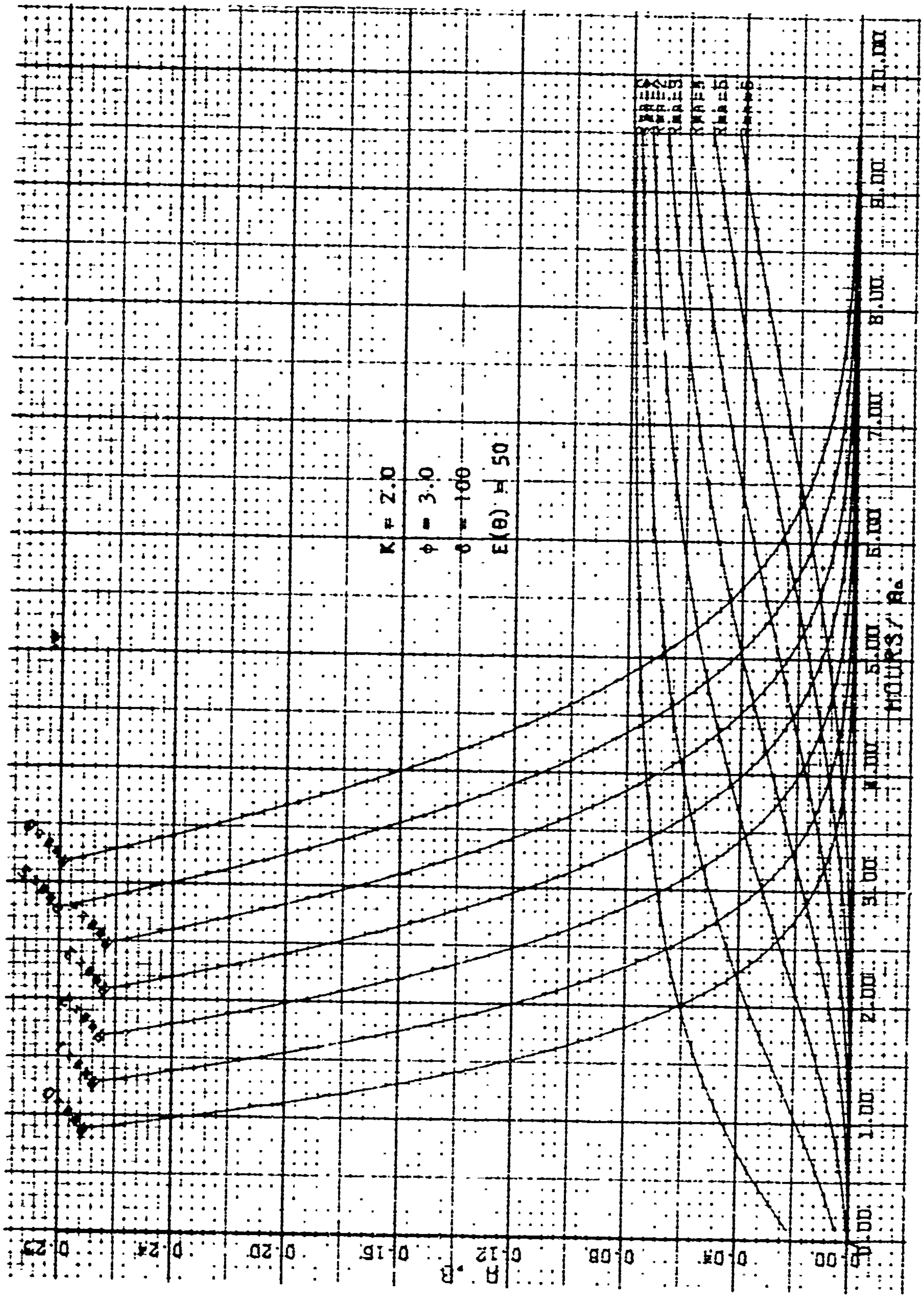

FIGURE 30 - Bayes Method; Criteria Set A-B

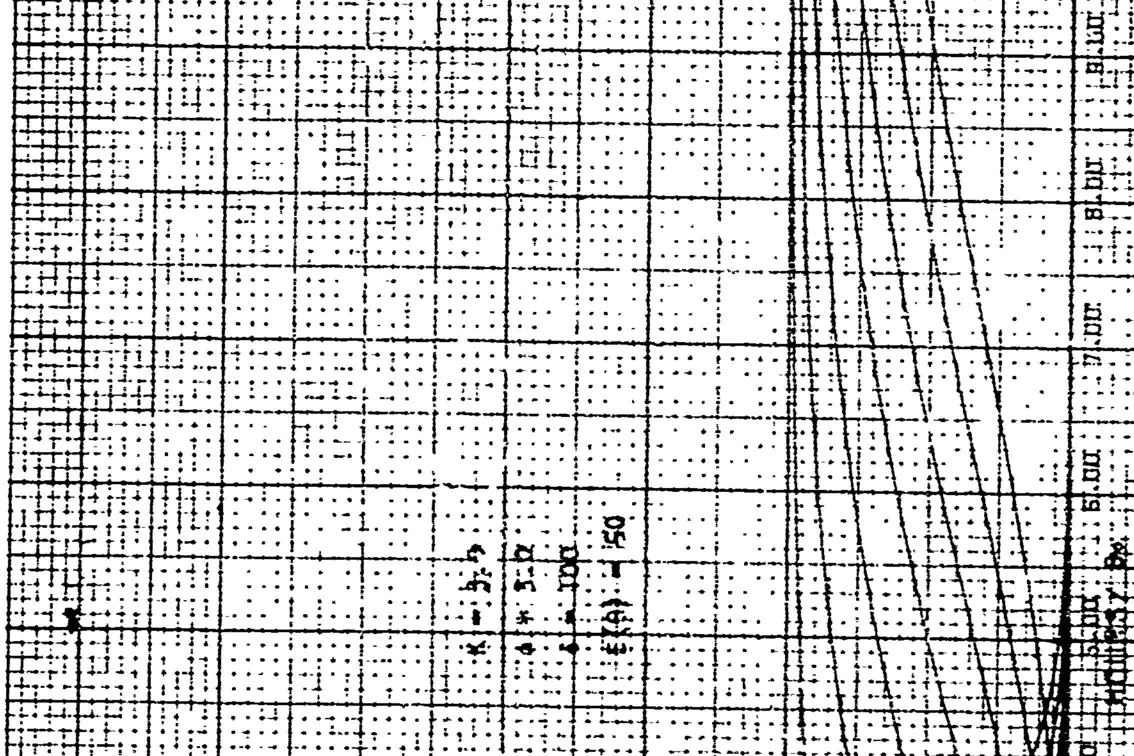

FIGURE 31 - Bayes Method; Criteria Set A-B

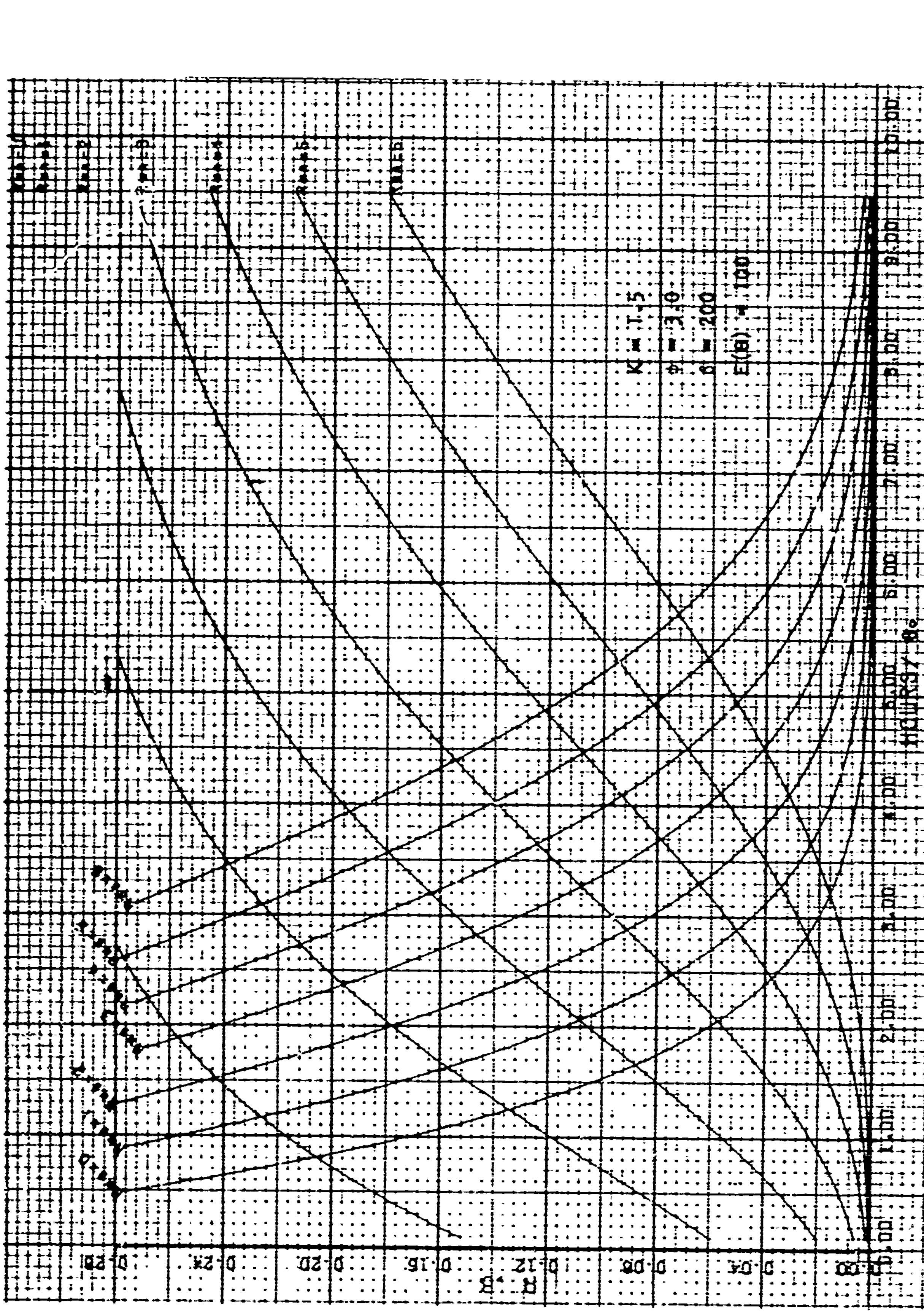

FIGURE 32 - Bayes Method; Criteria Set A-B

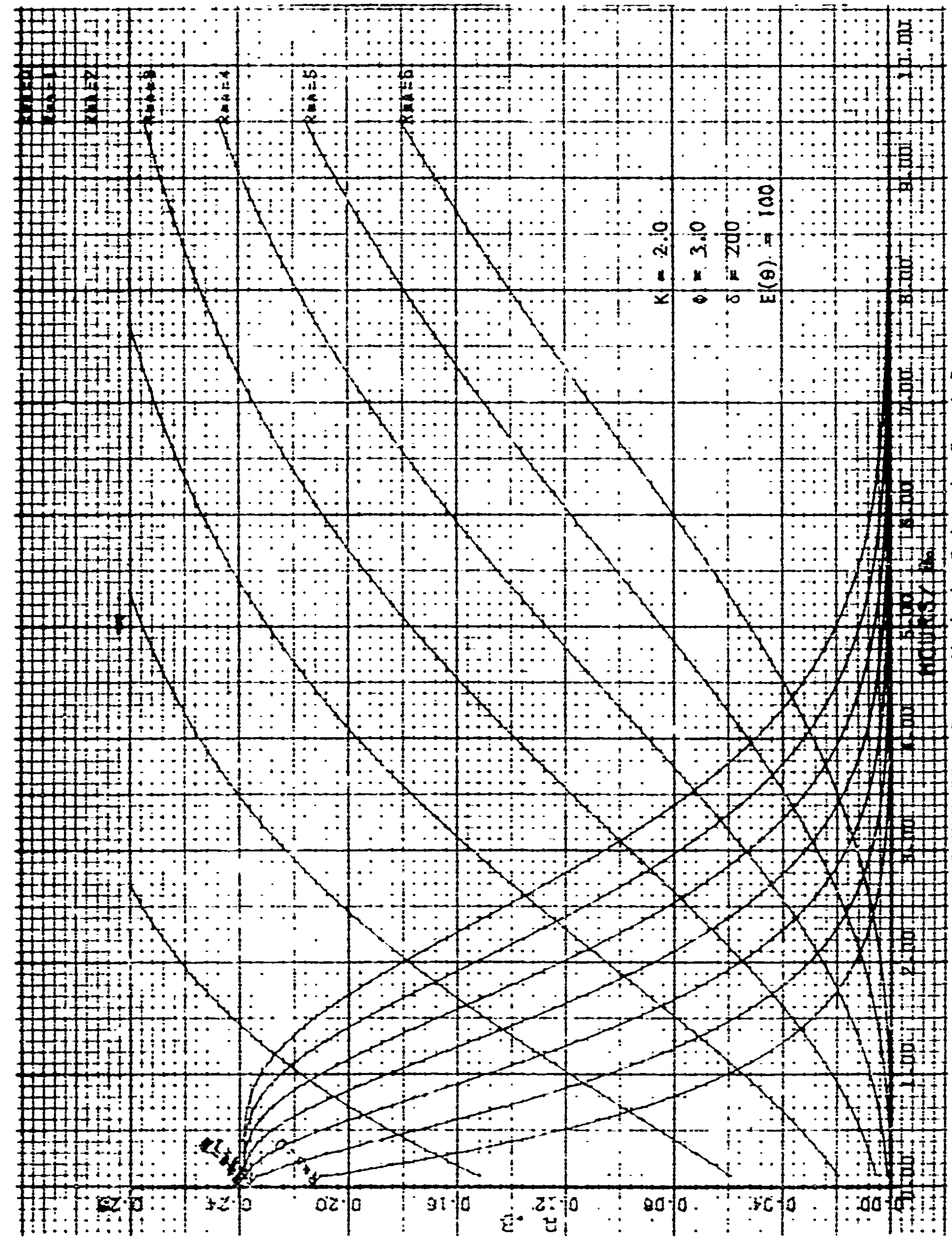

FIGURE 33 - Bayes Method; Criteria Set A-B

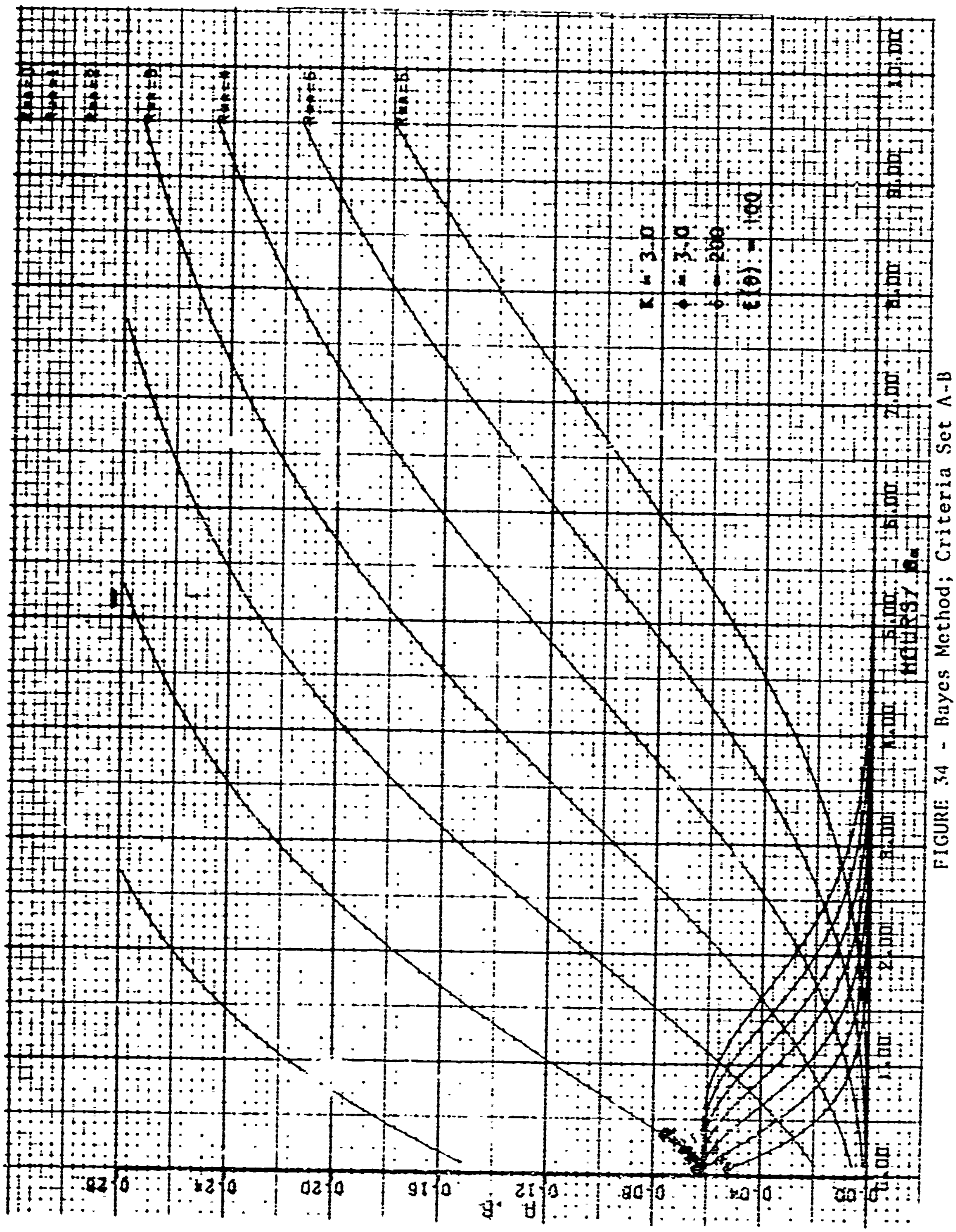

FIGURE 34 - Bayes Method; Criteria Set A-B

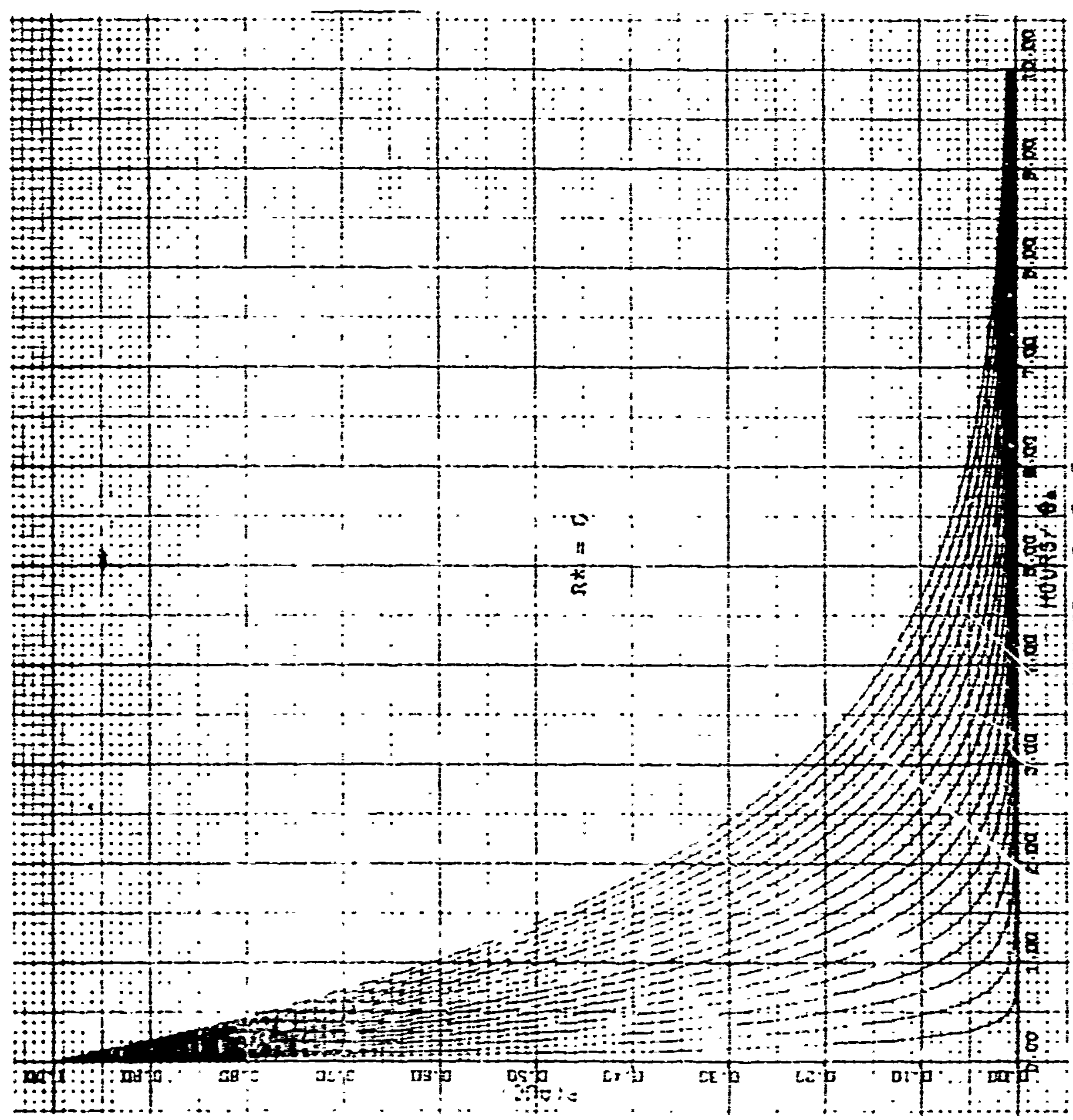

FIGURE 35 - O. C. Curve

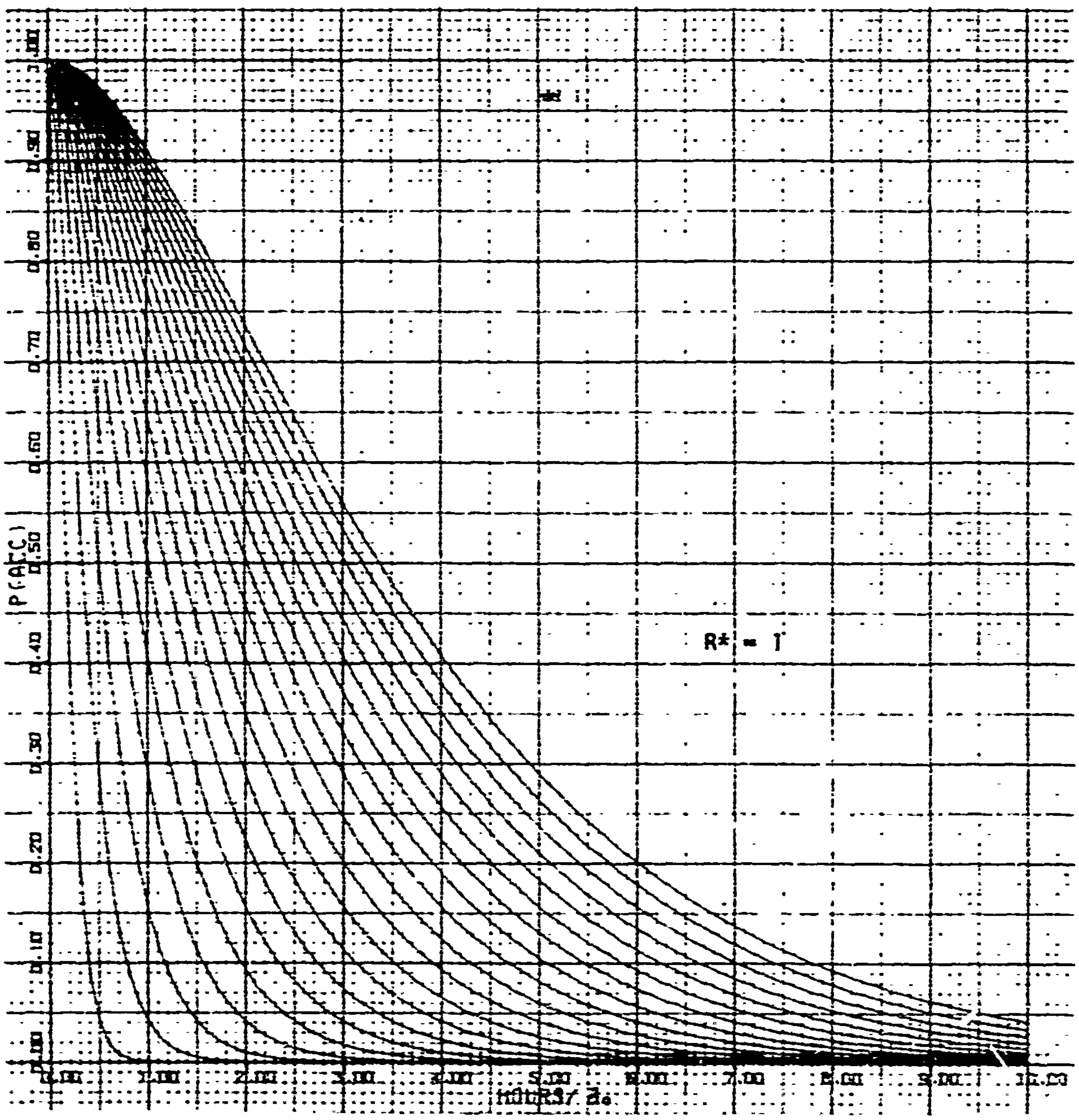

FIGURE 36 - O. C. Curve

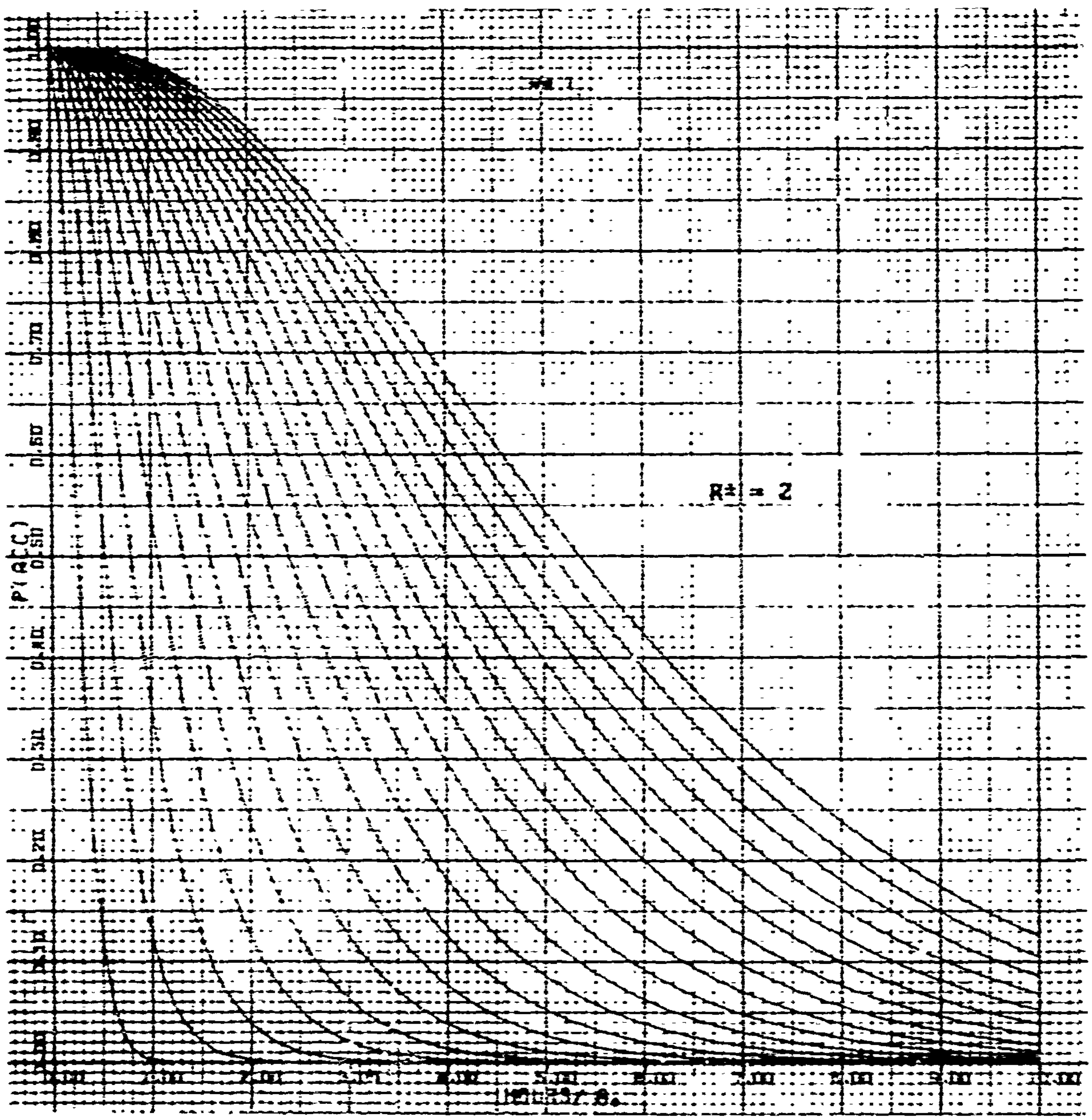

FIGURE 37 - O. C. Curve

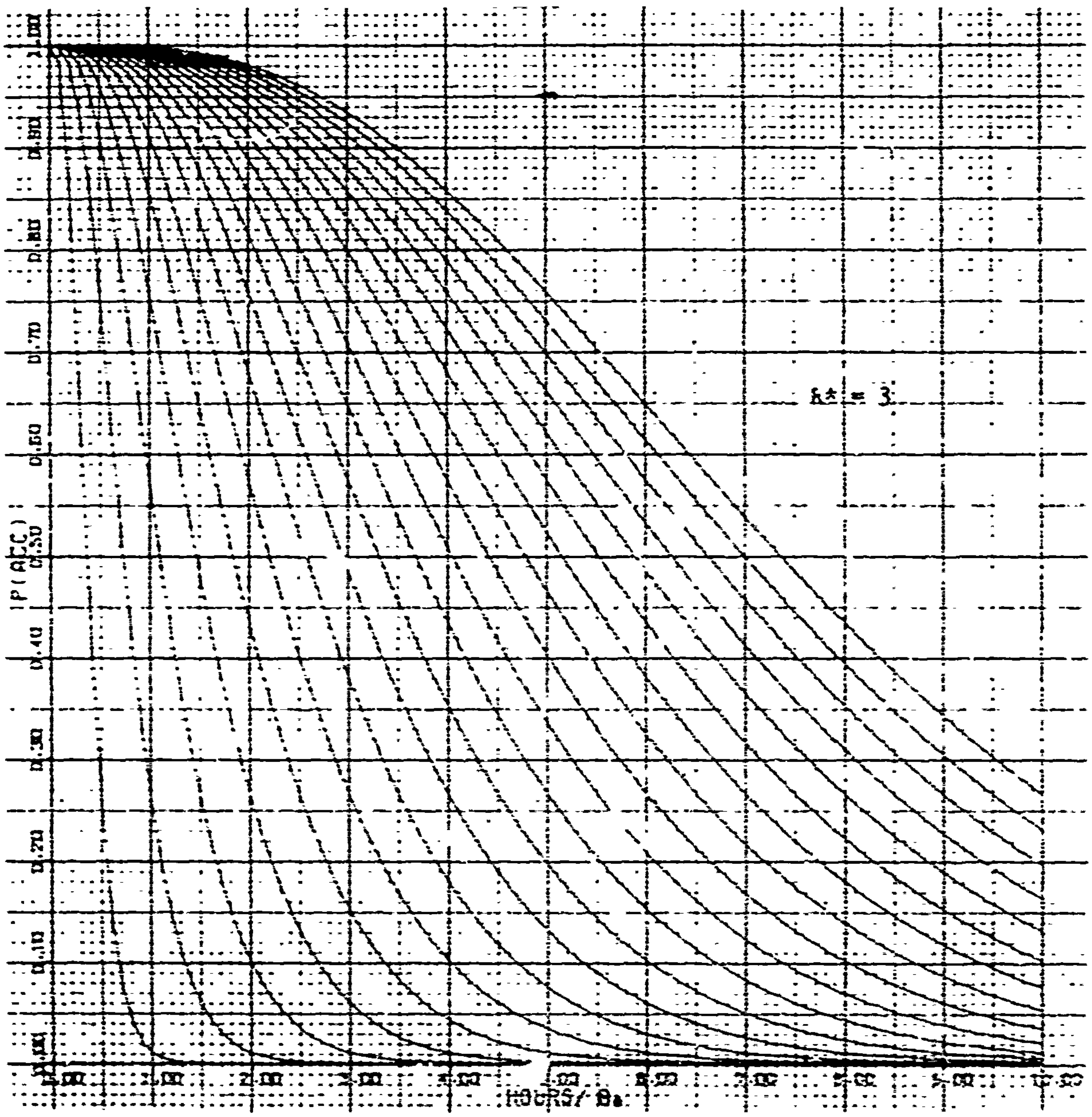

FIGURE 38 - O. C. Curve

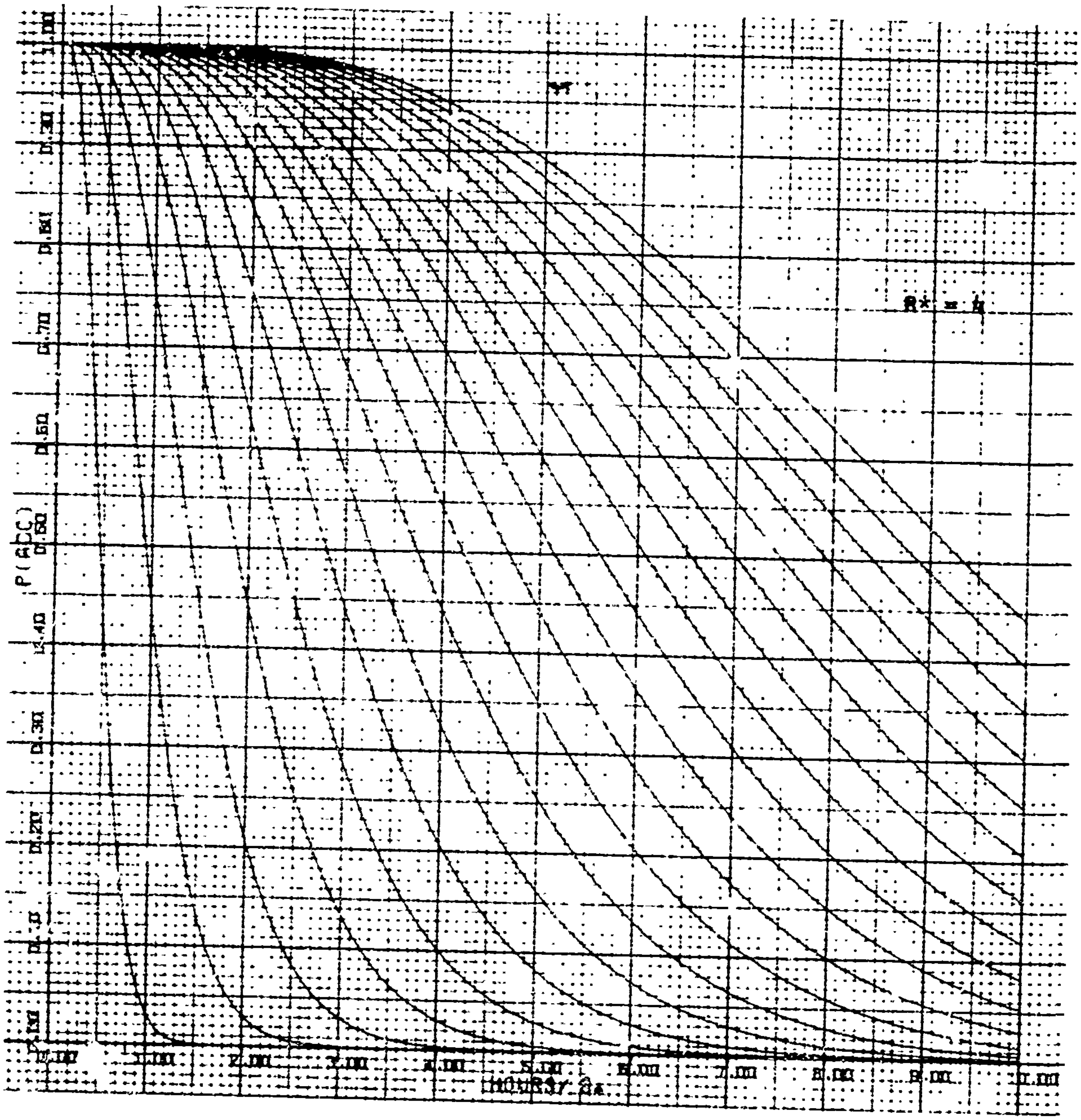

FIGURE 39 - O. C. Curve

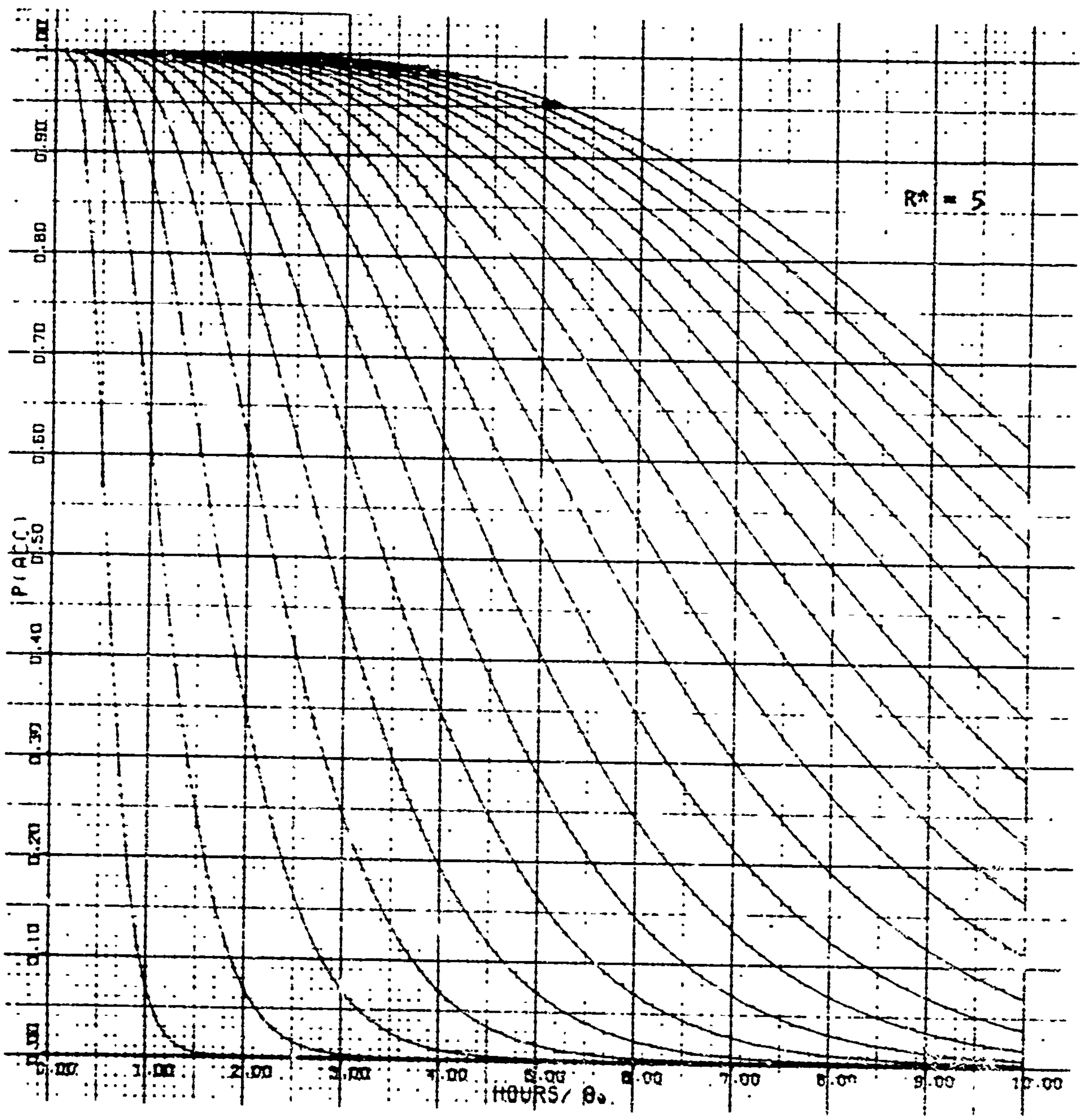

FIGURE 40 - O. C. Curve

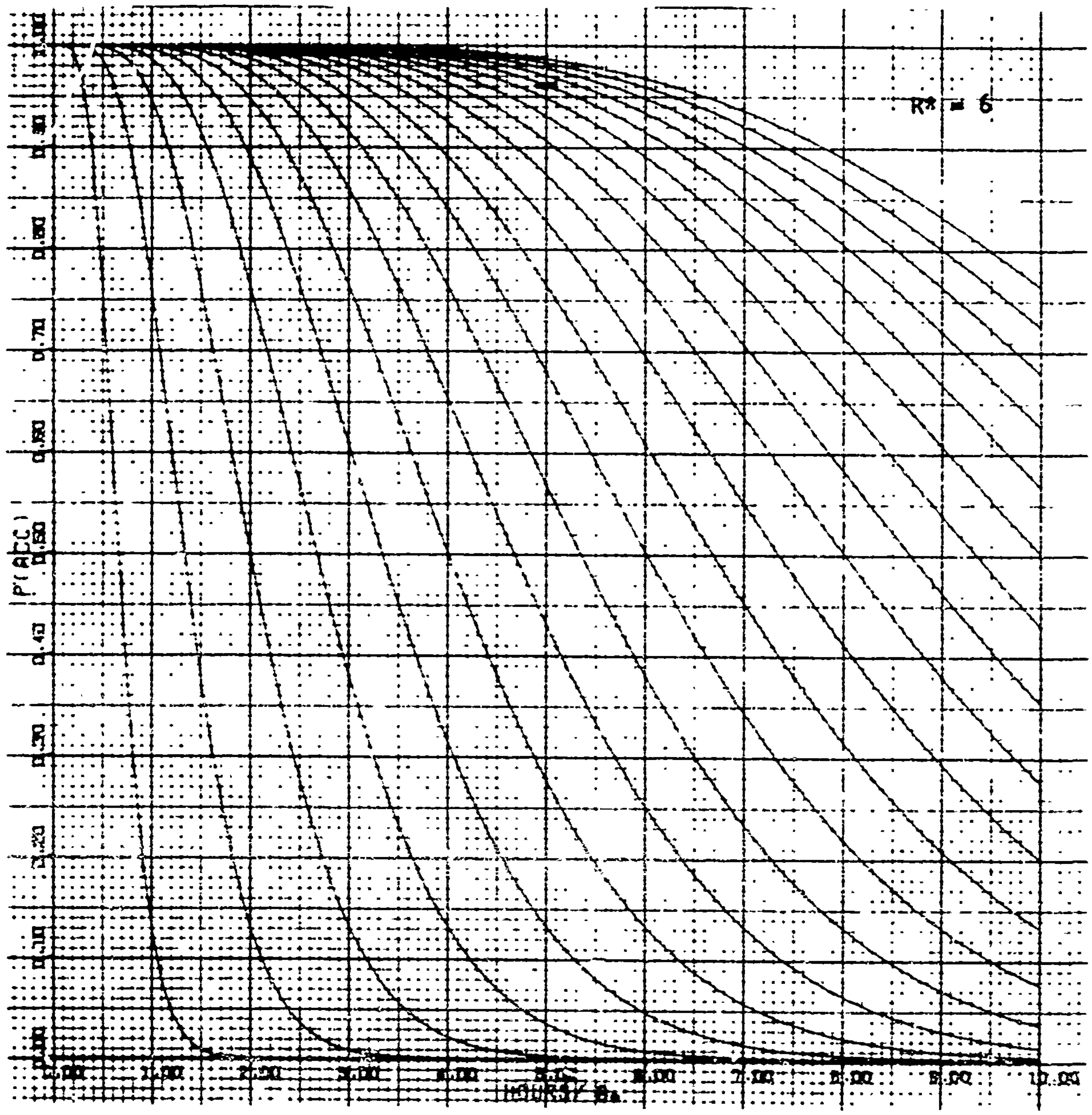

FIGURE 41 - O. C. Curve

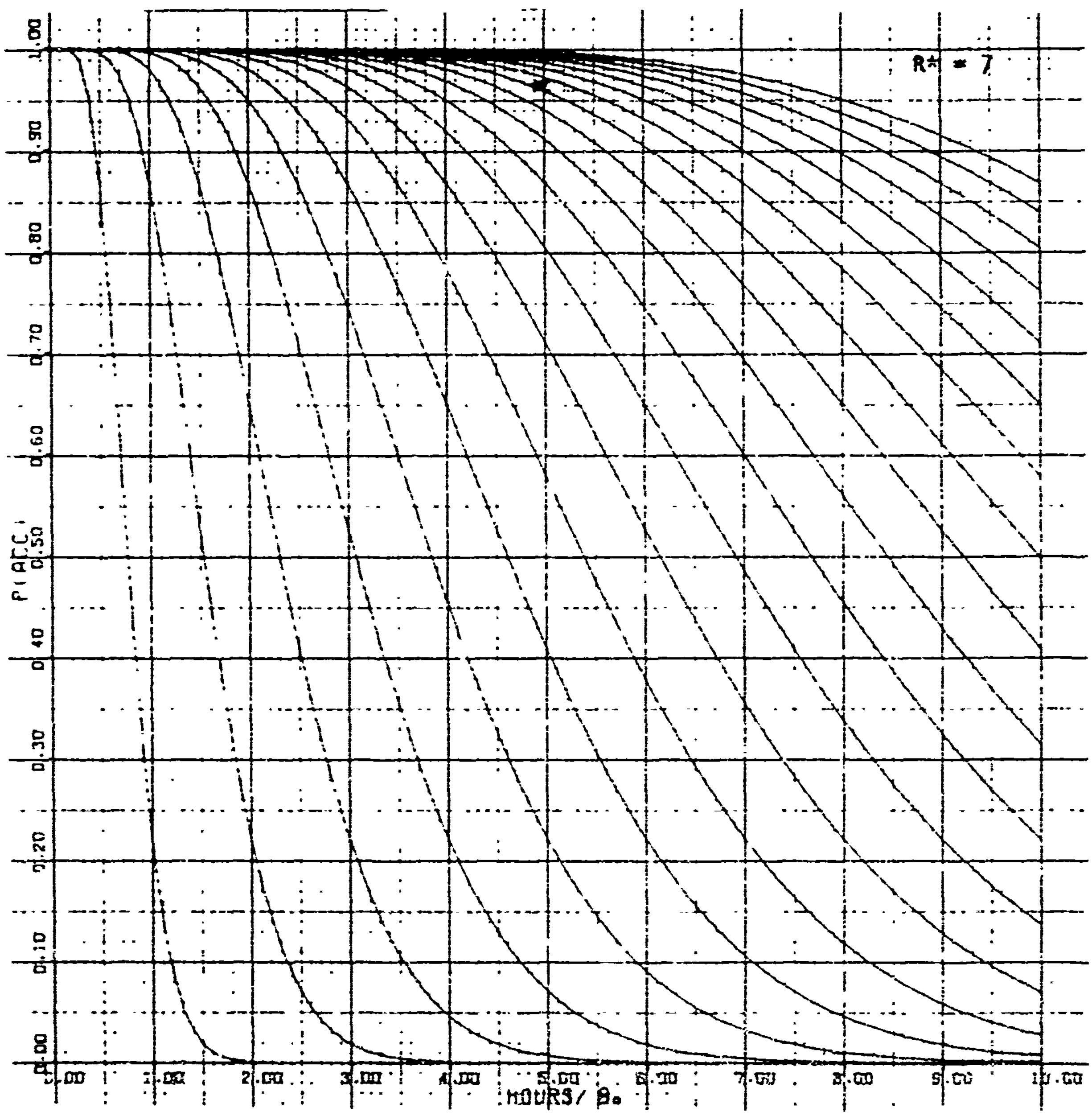

FIGURE 42 - O. C. Curve

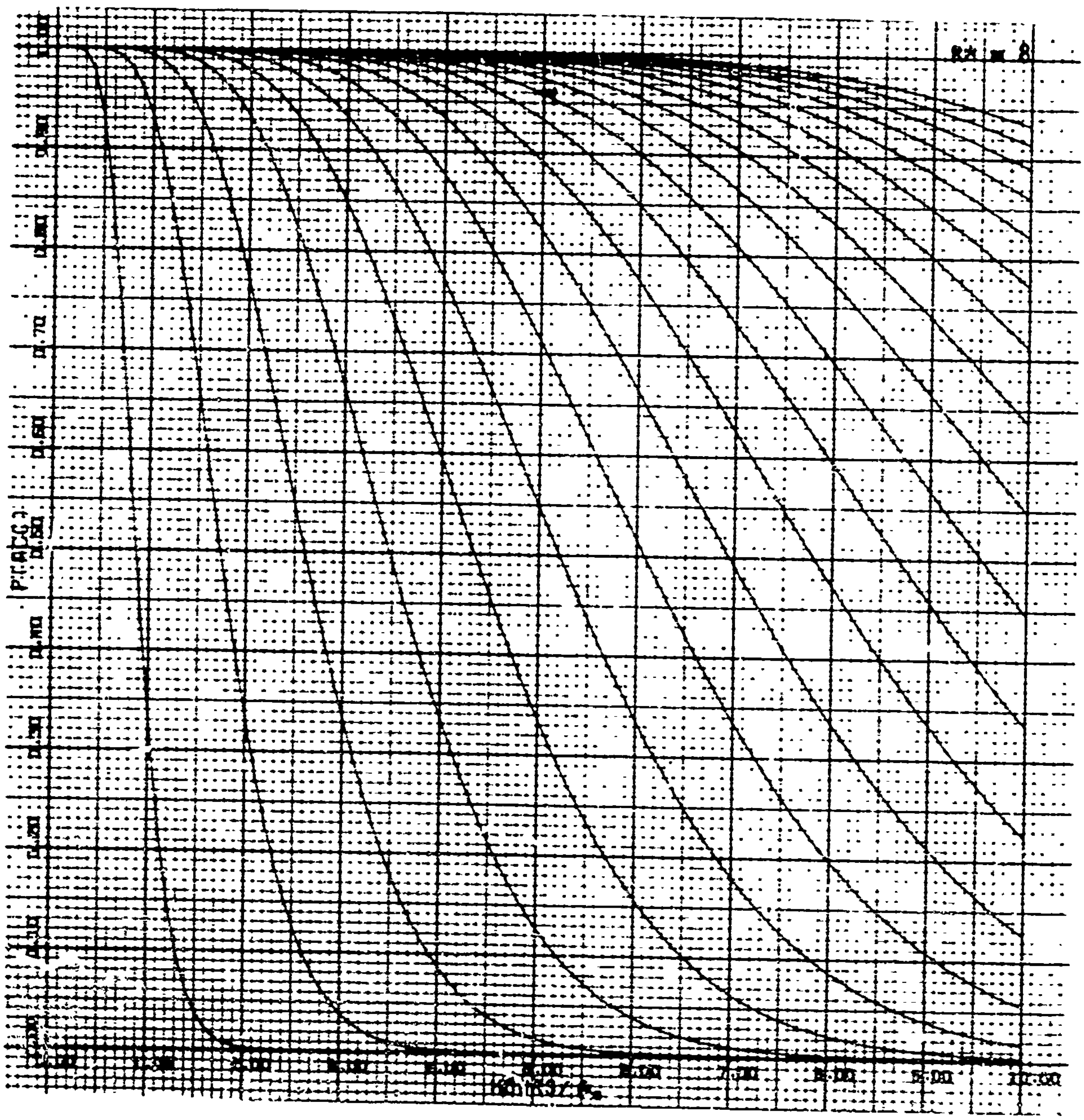

FIGURE 43 - O. C. Curve

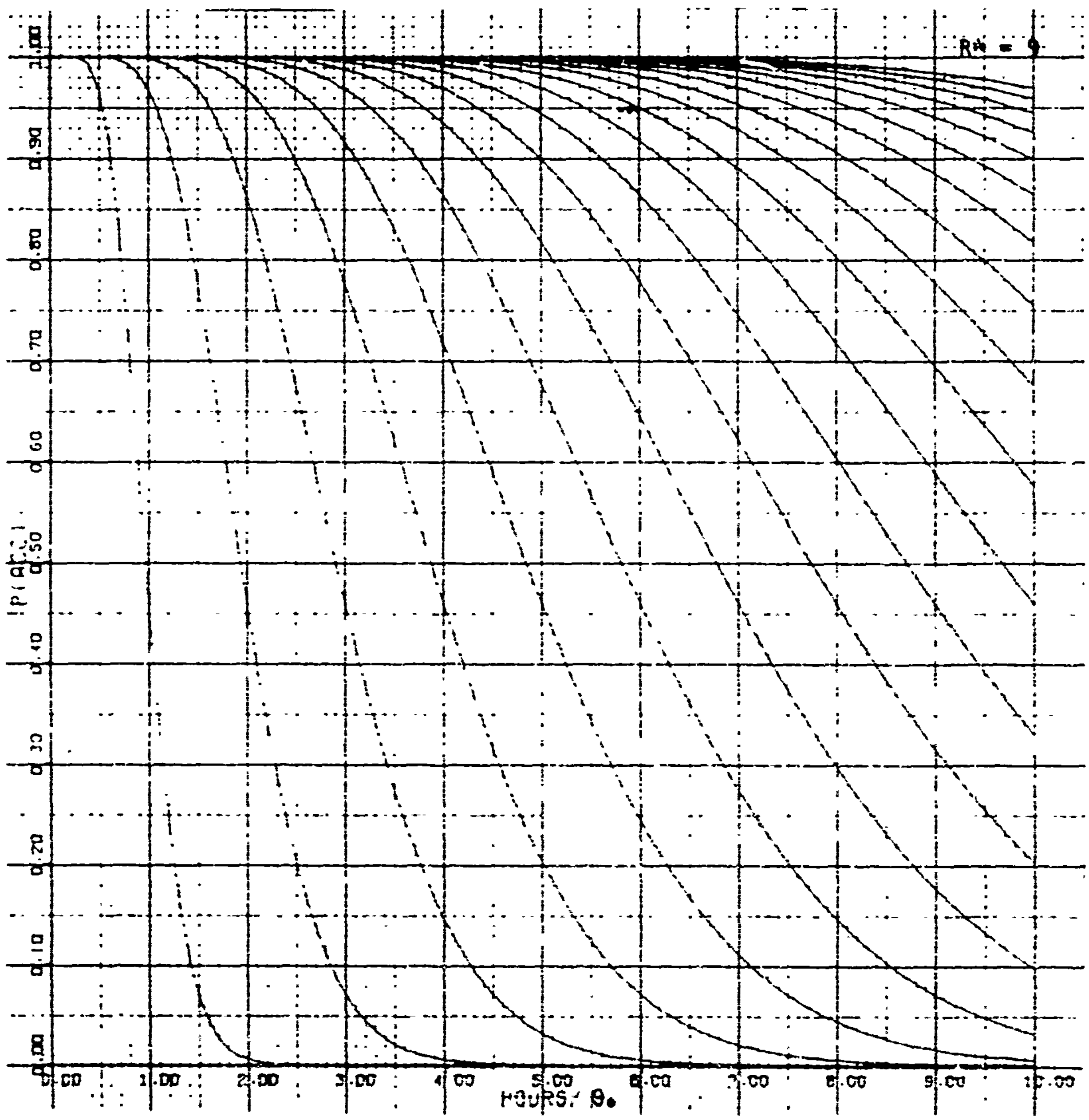

FIGURE 44 - O. C. Curve

APPENDIX A

BASIC FORMULAS

When the time to failure is exponentially distributed, the number of failures in a period of length T has the Poisson distribution with parameter T/θ, so that

$$P\left(Acc\Big|_{\theta}\right) = \sum_{r=0}^{r^*} \frac{\left(\frac{T}{\theta}\right)^r e^{-\frac{T}{\theta}}}{r!}$$

where $P(A_{cc/\theta})$ is the conditional probability of test acceptance (i.e., acceptance of the hypothesis that $\theta = \theta_0$) given that the MTBF equals θ, and r^* is the maximum number of failures permitted in the interal T, for an accept decision.

Classical test plan design is generally based on the following specifications:

A-1 $$P\left(Acc\Big|_{\theta=\theta_0}\right) = \sum_{r=0}^{r^*} \frac{\left(\frac{T}{\theta_0}\right)^r e^{-\frac{T}{\theta_0}}}{r!} \geq 1-\alpha$$

and

A-2 $$P\left(Acc\Big|_{\theta=\theta_1}\right) = \sum_{r=0}^{r^*} \frac{\left(\frac{T}{\theta_1}\right)^r e^{-\frac{T}{\theta_1}}}{r!} \leq \beta$$

where α, β designate the classical producer's, consumer's risk, respectively.

The Bayes risks are obtained by integrating $P(Acc|\theta)$ with the prior distribution $f(\theta)$. The expression for the Bayes posterior producer's risk, denoted as A, is obtained as follows:

by definition
$$A = P(\theta \geq \theta_0 | Rej) \equiv \frac{P(\theta \geq \theta_0 ; Rej)}{P(Rej)}$$

and
$$P(\theta \geq \theta_0 ; Rej) = \int_{\theta_0}^{\infty} \left(1 - P_{Acc|\theta}\right) f(\theta)\, d\theta$$

$$= P(\theta \geq \theta_0) - \int_{\theta_0}^{\infty} P(Acc|\theta) f(\theta)\, d\theta$$

$$= P(\theta \geq \theta_0) - P(Acc ; \theta \geq \theta_0)$$

where
$$P(Acc) = \int_0^{\infty} P(Acc|\theta) f(\theta)\, d\theta$$

and
$$P(Acc) + P(Rej) = 1$$

When the prior distribution, $f(\theta)$, is the inverted gamma density, it can be expressed as:

$$f(\theta) = \frac{\delta^{\phi}}{\Gamma(\phi)\theta^{\phi+1}} e^{-\frac{\delta}{\theta}} \qquad \theta \geq 0$$

$$= \quad 0 \qquad\qquad \theta < 0$$

where δ, ϕ denote the scale and shape parameters of this distribution, respectively, and are assumed to be known.

substituting the expressions for $f(\theta)$ and $P(Acc/\theta)$ into the previous equation yields

$$P(\theta \geq \theta_0; Rej) = \int_{\theta_0}^{\infty} \sum_{r=r^*+1}^{\infty} \frac{\delta^{\phi} e^{-\frac{\delta}{\theta}} \left(\frac{T}{\theta}\right)^{r} e^{-\frac{T}{\theta}}}{\Gamma(\phi)\theta^{\phi+1} r!} d\theta$$

combining terms gives

$$P(\theta \geq \theta_0; Rej) = \sum_{r=r^*+1}^{\infty} \frac{\delta^{\phi} T^{r}}{\Gamma(\phi) r!} \int_{\theta_0}^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta$$

the next several steps involve transformation of the integral to an incomplete gamma function by a change of variables, as follows:

let $\theta = x^{-1}$

$d\theta = -x^{-2} dx$

substituting these variables in the integral yields

$$\int_{\frac{1}{\theta_0}}^{0} \chi^{(\phi+r+1)} e^{-(T+\delta)\chi}\left(-\frac{1}{\chi^2}\right) d\chi$$

$$= \int_0^{\frac{1}{\theta_0}} \chi^{(\phi+r-1)} e^{-(T+\delta)\chi}\, d\chi$$

let $\quad y = (T+\delta)\chi$

substituting $\quad dy = (T+\delta)\, d\chi$

$$\int_0^{\left(\frac{T+\delta}{\theta_0}\right)} \left(\frac{y}{T+\delta}\right)^{\phi+r-1} e^{-y} \frac{dy}{(T+\delta)} = \frac{1}{(T+\delta)^{\phi+r}} \int_0^{\frac{T+\delta}{\theta_0}} y^{\phi+r-1} e^{-y}\, dy$$

The above integral, now in the form of the incomplete gamma function, can be solved in several different ways. Since tables of the incomplete gamma function are available, they may be used to obtain a solution; Deming (pg 471) cites several other procedures for evaluating this function. When ϕ is an integer > 0 the solution may be derived as follows, using operator notation:

$$\int = \frac{1}{D} = \frac{1}{D_1 + D_2}$$

and $$\int_0^{\frac{T+\delta}{\theta_0}} y^{\phi+r-1} e^{-y}\, dy = \frac{1}{D_1+D_2}\left[y^{\phi+r-1} e^{-y}\right]_0^{\frac{T+\delta}{\theta_0}}$$

but $$\frac{1}{D_1+D_2} = \frac{\frac{1}{D_2}}{1+\frac{D_1}{D_2}} = \frac{1}{D_2}\left[1-\left(\frac{D_1}{D_2}\right)^1+\left(\frac{D_1}{D_2}\right)^2-\left(\frac{D_1}{D_2}\right)^3+\cdots\right] =$$

performing the indicated operations gives

$$\left[\frac{1}{D_2}+(-1)^1\frac{D_1^1}{D_2^2}+(-1)^2\frac{D_1^2}{D_2^3}+\cdots+(-1)^{\phi+r-1}\frac{D_1^{\phi+r-1}}{D_2^{\phi+r}}+\cdots\right]\left[y^{\phi+r-1}e^{-y}\right]\Bigg|_0^{\frac{T+\delta}{\theta_0}}$$

$$\left[y^{\phi+r-1}\frac{1}{D_2}e^{-y}+(-1)^1D_1^1y^{\phi+r-1}\frac{1}{D_2^2}e^{-y}+(-1)^2D_1^2y^{\phi+r-1}\frac{1}{D_2^3}e^{-y}+\cdots\right.$$

$$\left.+(-1)^{\phi+r-1}D_1^{\phi+r-1}y^{\phi+r-1}\frac{1}{D_2^{\phi+r}}e^{-y}\right]\Bigg|_0^{\frac{T+\delta}{\theta_0}}=$$

$$\left[y^{\phi+r-1}\frac{e^{-y}}{(-1)^1}+(-1)^1(\phi+r-1)y^{\phi+r-2}\frac{e^{-y}}{(-1)^2}+(-1)^2(\phi+r-1)(\phi+r-2)y^{\phi+r-3}\frac{e^{-y}}{(-1)^3}\right.$$

$$\left.+\cdots+(-1)^{\phi+r-1}(\phi+r-1)(\phi+r-2)\cdots3\cdot2\cdot1\frac{e^{-y}}{(-1)^{\phi+r}}\right]\Bigg|_0^{\frac{T+\delta}{\theta_0}}=$$

$$e^{-\left(\frac{T+\delta}{\theta_0}\right)}\left[\frac{1}{(-1)^1}\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-1}+\frac{(-1)^1(\phi+r-1)}{(-1)^2}\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-2}+\frac{(-1)^2(\phi+r-1)(\phi+r-2)}{(-1)^3}\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-3}+\cdots\right.$$

$$\left.+\frac{(-1)^{\phi+r-1}}{(-1)^{\phi+r}}(\phi+r-1)(\phi+r-2)\cdots3\cdot2\cdot1\right]-1\left[0+\cdots+0+\frac{(-1)^{\phi+r-1}}{(-1)^{\phi+r}}(\phi+r-1)(\phi+r-2)\cdots3\cdot2\cdot1\right]=$$

$$-e^{-\left(\frac{T+\delta}{\theta_0}\right)}\left[\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-1}+(\phi+r-1)\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-2}+(\phi+r-1)(\phi+r-2)\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-3}+\cdots\right.$$

$$\left.+(\phi+r-1)(\phi+r-2)\cdots 3\cdot 2\cdot 1\right]+(\phi+r-1)(\phi+r-2)\cdots 3\cdot 2\cdot 1 =$$

$$\left\{-e^{-\left(\frac{T+\delta}{\theta_0}\right)}\left[\frac{1}{(\phi+r-1)!}\left(\frac{T+\delta}{\theta_0}\right)^{\phi+r-1}+\cdots+\frac{1}{1!}\left(\frac{T+\delta}{\theta_0}\right)^{1}+\frac{1}{0!}\right]+1\right\}(\phi+r-1)! =$$

let $\frac{T+\delta}{\theta_0}=q_0$

$$\left\{-e^{-q_0}\left[\sum_{i=1}^{i=\phi+r}\frac{1}{(\phi+r-i)!}\,q_0^{\phi+r-i}\right]+1\right\}(\phi+r-1)! =$$

$$(\phi+r-1)!\left[1-\sum_{i=1}^{\phi+r}\frac{e^{-q_0}q_0^{\phi+r-i}}{(\phi+r-i)!}\right]=(\phi+r-1)!\left[1-\sum_{j=\phi+r-1}^{j=0}\frac{e^{-q_0}q_0^{j}}{j!}\right]=(\phi+r-1)!\left[1-\sum_{j=0}^{j=\phi+r-1}\frac{e^{-q_0}q_0^{j}}{j!}\right]$$

Let $j=\phi+r-i$

but $$1-\sum_{j=0}^{j=\phi+r-1}\frac{e^{-q_0}q_0^{j}}{j!}=\sum_{j=\phi+r}^{\infty}\frac{e^{-q_0}q_0^{j}}{j!}$$

Therefore, the equivalent expression is

$$(\phi+r-1)!\sum_{j=\phi+r}^{\infty}\frac{e^{-q_o}q_o^{\,j}}{j!}$$

the entire function can now be written as

$$P(\theta\geq\theta_{oi};\,Rej)=\sum_{r=r^*+1}^{\infty}\frac{\delta^{\phi}T^{r}(\phi+r-1)!}{(\phi-1)!\,r!\,(T+\delta)^{\phi+r}}\left[\sum_{j=\phi+r}^{\infty}\frac{e^{-q_o}q_o^{\,j}}{j!}\right]$$

this expression can be further simplified by substituting

$$P=\frac{T}{T+\delta}\qquad 1-P=\frac{\delta}{T+\delta}$$

to yield

A-3 $$P(\theta\geq\theta_o;\,Rej)=e^{-q_o}(1-P)^{\phi}\sum_{r=r^*+1}^{\infty}\sum_{j=\phi+r}^{\infty}\binom{\phi+r-1}{r}\frac{P^r q_o^{\,j}}{j!}$$

and

A-4 $$P(Rej)=\sum_{r=r^*+1}^{\infty}\frac{(\phi+r-1)!}{(\phi-1)!\,r!}P^r(1-P)^{\phi}$$

also $$P(Acc)=1-P(Rej)$$

A-5 $$=\sum_{r=0}^{r^*}\frac{(\phi+r-1)!}{(\phi-1)!\,r!}P^r(1-P)^{\phi}$$

By definition, the Bayes consumer's risk B is expressed as:

$$B = \frac{P(\theta \le \theta_1 ; Acc)}{P(Acc)}$$

$$P(\theta \le \theta_1 ; Acc) = \sum_{r=0}^{r^*} \frac{\delta^{\phi} T^{r}}{\Gamma(\phi)\, r!} \int_0^{\theta_1} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta$$

it has previously been shown (derivation of equation 3) that

$$\int_{\theta_0}^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta = \frac{1}{(T+\delta)^{\phi+r}} \int_0^{\frac{T+\delta}{\theta_0}} y^{\phi+r-1} e^{-y} dy$$

and for : = integer, that:

$$\int_{\theta_0}^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta = \frac{1}{(T+\delta)^{\phi+r}} \left[(\phi+r-1)! \left(1 - \sum_{j=0}^{\phi+r-1} \frac{e^{-q_0} q_0^{j}}{j!} \right) \right]$$

noting that

$$\int_0^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta = \int_0^{\theta_1} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta$$

$$+ \int_{\theta_1}^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta$$

$$\int_0^{\theta_1} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta = \int_0^{\infty} \theta^{-(\phi+r+1)} e^{-\left(\frac{\delta+T}{\theta}\right)} d\theta - \frac{1}{(T+\delta)^{\phi+r}} \int_0^{\frac{T+\delta}{\theta_1}} y^{(\phi+r-1)} e^{-y} dy$$

$$= \frac{1}{(T+\delta)^{\phi+r}} \left[\int_0^{\infty} y^{(\phi+r-1)} e^{-y} dy - \int_0^{\frac{T+\delta}{\theta_1}} y^{(\phi+r-1)} e^{-y} dy \right]$$

$$= \frac{1}{(T+\delta)^{\phi+r}} \left[\Gamma(\phi+r) - \Gamma_{\frac{T+\delta}{\theta_1}}(\phi+r) \right]$$

if ϕ is an integer, this expression further reduces to

$$= \frac{(\phi+r-1)!}{(T+\delta)^{\phi+r}} \left[1 - \left(1 - \sum_{j=0}^{\phi+r-1} \frac{e^{-q_1} q_1^{\,j}}{j!} \right) \right]$$

where

$$q_1 = \frac{T+\delta}{\theta_1}$$

simplifying

$$= \frac{(\phi+r-1)!}{(T+\delta)^{\phi+r}} \sum_{j=0}^{\phi+r-1} \frac{e^{-q_1} q_1^{\,j}}{j!}$$

Therefore

$$P(\theta \leq \theta_1; Acc) = \sum_{r=0}^{r^*} \frac{\delta^{\phi} T^r (\phi+r-1)!}{(\phi-1)!\, r!\, (T+\delta)^{\phi+r}} \left[\sum_{j=0}^{\phi+r-1} \frac{e^{-g_1} g_1^j}{j!} \right]$$

A-6
$$= \sum_{r=0}^{r^*} \frac{(\phi+r-1)!}{(\phi-1)!\, r!}\, p^r (1-p)^{\phi} \left[\sum_{j=0}^{\phi+r-1} \frac{e^{-g_1} g_1^j}{j!} \right]$$

the equation for $P(\theta \geq \theta_0)$ is determined as follows:

$$P(\theta \geq \theta_0) = \int_{\theta_0}^{\infty} f(\theta)\, d\theta$$

$$= \int_{\theta_0}^{\infty} \frac{\delta^{\phi}}{\Gamma(\phi)\theta^{\phi+1}} e^{-\frac{\delta}{\theta}} d\theta$$

$$= \frac{\delta^{\phi}}{\Gamma(\phi)} \int_{\theta_0}^{\infty} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta$$

this integral is similar to the one appearing in the derivation of equation A-4. By anology;

$$\int_{\theta_0}^{\infty} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta = \frac{1}{\delta^{\phi}} \Gamma_{\frac{\delta}{\theta_0}}(\phi)$$

$$P(\theta > \theta_0) = \frac{\Gamma_{\frac{\delta}{\theta}}(\phi)}{\Gamma(\phi)}$$

for ϕ = integer, this equation reduces to

A-7 $$P(\theta > \theta_0) = 1 - \sum_{j=0}^{\phi-1} \frac{e^{-q} q^j}{j!} = \sum_{j=\phi}^{\infty} \frac{e^{-q} q^j}{j!}$$

where $q = \frac{\delta}{\theta_0}$

The procedure for evaluating $P(\theta \leq \theta_1)$ is:

$$P(\theta \leq \theta_1) = \int_0^{\theta_1} f(\theta) d\theta = \int_0^{\theta_1} \frac{\delta^{\phi}}{\Gamma(\phi)\theta^{(\phi+1)}} e^{-\frac{\delta}{\theta}} d\theta$$

$$= \frac{\delta^{\phi}}{\Gamma(\phi)} \int_0^{\theta_1} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta$$

but

$$\int_0^{\theta_1} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta = \int_0^{\infty} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta - \int_{\theta_1}^{\infty} \theta^{-(\phi+1)} e^{-\frac{\delta}{\theta}} d\theta$$

$$= \frac{1}{\delta^{\phi}} \left[\Gamma(\phi) - \Gamma_{\frac{\delta}{\theta_1}}(\phi) \right]$$

as shown in the derivation leading to equation A-6.

Again, if ϕ is integer valued

thus $$P(\theta \leq \theta_1) = \sum_{j=0}^{\phi-1} \frac{e^{-q'}(q')^j}{j!}$$

where $$q' = \frac{\delta}{\theta_1}$$

equations A-1 through A-8 are the expressions used, in various combinations, to compute the risk functions corresponding to the values assigned to T/θ_o and r*. These data were then incorporated into a plotting routine which generated all the graphs shown in the body of the report. To illustrate, graphs of the risk criteria set A - B were formulated using the following relationships:

$$A = \frac{\text{Equation (A - 3)}}{\text{Equation (A - 4)}}$$

$$B = \frac{\text{Equation (A - 6)}}{\text{Equation (A - 5)}}$$

for criteria set A_1-B_1, the following equations apply

$$A_1 = \frac{\text{Equation (A - 3)}}{\text{Equation (A - 7)}}$$

$$B_1 = \frac{\text{Equation (A - 6)}}{\text{Equation (A - 8)}}$$

for criteria set A-B_1

$$A = \frac{\text{Equation (A - 3)}}{\text{Equation (A - 4)}}$$

$$B = \text{Equation (A - 2)}$$